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Theory of a Nearly Two-Dimensional Dipolar Bose Gas

by

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UNITED STATES NAVAL ACADEMY
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Abstract

This project develops a theoretical model for gases of bosonic atoms at ultracold, but finite temperatures. Under these circumstances, the gas can undergo a phase transition to a purely quantum mechanical state, a Bose-Einstein condensate (BEC), where the atoms cease to behave like distinguishable entities, and instead form a single macroscopic matter wave. At exactly zero temperature, all of the atoms occupy the BEC; at finite temperatures, a significant fraction of the atoms leave the BEC and form a thermal cloud. Thus, the state of a low, but finite-temperature gas of bosonic atoms involves the coexistence of a BEC and a thermal cloud. Further, the atoms can interact in a variety of different ways, which have important consequences for the state of the gas. We consider both short range contact interactions and dipolar interactions, where the atoms interact via the long-range, anisotropic dipole-dipole force. We develop this model in both three- and two-dimensional geometries.

When the dipolar gas is confined to a nearly two-dimensional geometry, the BEC can exhibit an unusual excitation spectrum, where certain excitations that would usually be highly energetic are instead only weakly energetic. These excitations are called rotons. Because it costs minimal energy for an atom to leave the BEC and be promoted to a roton excitation, modest temperatures can result in fairly large thermal cloud occupations. This project describes the relative occupations of the BEC and the thermal cloud at given temperatures, and when roton excitations are present. Further, BECs in nearly 2D geometries take the form of quasi-condensates, or BECs with finite spatial extent. Quasi-condensates behave like BECs on shorter length scales, but not on longer length scales. The project incorporates the presence of a quasi-condensate into the theoretical model, to more accurately describe the state of the system at hand.

Keywords: Bose Einstein condensation, ultra cold physics, condensed matter, dipoles

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Contents

I. BACKGROUND

Quantum mechanics, the science of tiny objects, plays an increasingly significant role in modern technologies, and is of inherent interest due to its many peculiarities. For example, quantum mechanics predicts that atoms possess wave-like properties, such as a wavelength, and can be classified as either "fermions" or "bosons." Thus, every particle has a wavelength, yet also behaves like an individual particle with either bosonic or fermionic properties. Additionally, both wave and particle treatments are necessary to fully understand the behavior of quantum objects. Concerning the particle classifications, examples of fermions are the matter-particles that everyone learns about in chemistry class: neutrons, protons, and electrons. Bosons are generally seen as force-carrier particles, or particles that propagate the four fundamental forces (photons are bosons, for example). However, certain atoms, such as Helium-4 (^{4}He), which is composed entirely of fermions, acts like a bosons. It turns out that this is the case whenever an object like an atom is composed of an even number of fermionic constituents. These atoms are called "boson-like," [1], though here we will simply refer to them as bosons. Whereas fermions are not allowed to occupy the same point in space due to the Pauli Exclusion Principle, bosons can occupy the same point in space. This is the major distinction between them. Quantum mechanically speaking, this corresponds to fermions not being able to occupy the same quantum state, with bosons being able to occupy the same quantum state. In fact, an unlimited number of bosons can occupy a given quantum state. If electrons were bosons, they could then occupy the same orbitals in atoms, and the chemistry of our world would certainly be a lot different.

The wavelength of atoms is qualitatively determined by the De Broglie wavelength λ, which is inversely proportional to the atom's momentum p, $\lambda = \hbar/p$, where $\hbar = 1.055 \times 10^{-34}$ J $\cdot$ s is the reduced Planck constant. In a gas, only when the inter-particle spacing approaches the order of the de Broglie wavelength do quantum mechanical effects begin to matter. Thus, at relatively high temperatures when the average momentum of the atoms is large, the properties of an atomic gas do not depend on whether the atoms are fermions or bosons; i.e. the high temperature gas is classical. As the temperature drops to the ultra-cold regime, the kinetic energy each particle has decreases, and the particle wavelengths grow and begin to overlap. If the atoms are bosons, they form a Bose-Einstein condensate (BEC). The atoms can no longer be described individually and instead behave like a giant matter

wave.

The Bose-Einstein condensate received its name from two instrumental physicists and mathematicians that heavily contributed to its finding. Satyendra Nath Bose earned a bachelor's and a master's degree at the age of twenty and then began translating Albert Einstein's papers on general and special relativity in order to better understand relativity. As he progressed in his studies, he began to delve into quantum mechanics more. Eventually, Bose's contributions to quantum physics would be so significant Paul Dirac, one of the greatest physicists of the 20th century, would name the boson particle in his honor. In 1924, Bose derived Planck's quantum radiation law with no reference to classical physics. In order to be published, he sent the paper to Einstein to translate it.

The other contributing scientist is world famous physicist Albert Einstein, maybe best known for his mass-energy equivalence equation, but he made many more extensive contributions to physics. His works include, but are not limited to special relativity, general relativity, evolving quantum theory, and the explanation of the photoelectric effect. However, his significant contribution to Bose-Einstein Condensates lies in response to receiving Bose's work on quantum statistics of light quanta (photons).

Einstein thoroughly enjoyed the paper so he used his influence to publish the paper and personally translated it to German. He then extended Bose's idea to atoms, which led to the creation of Bose-Einstein statistics. Bosons are indistinguishable from each other and can occupy the same energy levels. Therefore, we count indistinguishable bosons and fermions much differently. Indistinguishable versus distinguishable statistics can be eloquently shown through coin flipping. Say we want to count the probability of flipping two coins and getting a heads (H) and a tails (T). Common sense dictates that there are four possibilities (or microstates), HH, HT, TH, and TT because you can individually identify which coin is which. Maybe the two coins are a nickel and a quarter or quarters of different years. The important part is that you can distinguish between them and therefore can distinguish separate microstates. With these distinguishable coins, a tails and heads combination would have two out of four total microstates or a $\frac{1}{2}$ chance of occurring. If the coins were indistinguishable, the statistics change. Consider taking two newly minted identical quarters. Since there is no possible way to tell the coins apart, HT and TH are the same microstate. Because of this, the microstates would be HH, HT, and TT. HT would have $\frac{1}{3}$ chance of occurring. This difference accounts for the seemingly innocuous, but crucial difference in the statistics used to

analyze bosons. Following are the different equations for Maxwell-Boltzmann, Fermi-Dirac and Bose-Einstein statistics

$$\text{Maxwell} - \text{Boltzmann}: \quad n_i = \frac{g_i}{e^{(\epsilon_i - \mu)/K_b T}} \tag{1}$$

$$\text{Fermi} - \text{Dirac}: \quad n_i = \frac{g_i}{e^{(\epsilon_i - \mu)/k_b T} + 1} \tag{2}$$

$$\text{Bose} - \text{Einstein}: \quad n_i = \frac{g_i}{e^{(\epsilon_i - \mu)/k_b T} - 1} \tag{3}$$

In Eq 1, Eq 2, and Eq 3, n_i describes the number of particles in a given state, i. The state is given by the energy level associated with it ϵ_i. k_b is the Boltzmann constant and T stands for the temperature at which the state is being measured. μ is a Lagrange multiplier used to fix particle number. If we are modeling photons, we do not require constant particle numbers so we can ignore μ. Finally, g_i is the degeneracy associated with the selected state. If two states, ϵ_2 and ϵ_3 share the same energy level, ϵ_2, the states are called degenerate and g_i will account for it.

Equation 1, Maxwell-Boltzmann statistics, describes the distribution of non-interacting particles in thermal equilibrium at temperatures high enough (or density low enough) to ignore quantum effects. We use these statistics to model a gas (whether indistinguishable or distinguishable) at high temperatures where the gas behaves classically. Equation 2, Fermi-Dirac statistics, describes the distribution of indistinguishable fermions, particles that obey the Pauli-Exclusion Principle. These statistics cover the quantum realm, but specifically with regards to fermions. As previously stated, the inability of fermions to occupy the same quantum space requires we treat them differently. Finally, equation 3, Bose Einstein statistics, describes the energy level occupations of indistinguishable bosons. [2] These statistics cover the quantum realm with regards to bosons.

In 1924, Einstein predicted, based purely on Eq. 3 and Bose-Einstein statistics, a unique phenomenon occurs at low temperature. [3] As temperature decreases, a particle will lose energy, leaking it to the environment. Near absolute zero, this particle will most likely occupy its ground state, or least energetic state. When a collection of bosons all occupy the

ground state simultaneously, a new state of matter called the BEC appears. In Einstein's era, however, there was no way to cool anything to the temperatures required to observe physical BECs, so physical exploration was tabled until adequate cooling techniques existed.

As time progressed, more research could be made towards the ultimate goal of creating this new state of matter. Superfluidity was first observed in liquid Helium-4 in 1937, at temperatures below T=2.17 Kelvin. In 1938, Fritz London proposed BECs as a mechanism for superfluidity. Superfluidity, possibly the most interesting trait of BECs, is when a liquid/current flows with absolutely zero viscosity below a certain critical temperature. [4] Interestingly, this is the underlying mechanism for superconductivity in many metal and ceramic materials. If two electrons pair together, they act like a boson. Once you have a collection of bosons (electron pairs), they can condense and achieve superfluid properties. However, you need multiple bosons to achieve a BEC. A single boson cannot become a BEC, but many pairs of electrons will be able to condense. It is believed that the superfluidity in liquid Helium-4 is due to the presence of a BEC in the liquid, though this is difficult to show directly. However, not all BECs produce superfluids, and not all superfluidity stems from BECs. Superfluidity remains desirable, because it would eliminate energy losses in fluid systems and electrical systems.

Despite continued predictions, Bose-Einstein Condensates remained elusive due to the technological restraint of lowering temperatures down to near absolute zero. In 1978, laser cooling was invented and the ability to reach previously unfathomable temperatures, within nanoKelvin of absolute zero, became realistic. As the technology developed, this type of cooling became readily available and practical. Laser cooling is successful because photons have momentum. If a photon is traveling in the opposite direction of an atom and is absorbed, the atom will slow (Imagine a collision between a bowling ball rolling down an alley and a ping pong ball shot in the opposite direction. The bowling ball slows, but not significantly enough to be noticed). Then, the atom emits the photon in a random direction. On average, this will additionally decrease the momentum of the atom. This can be imagined as the bowling ball absorbed the ping pong ball and randomly ejecting it. In half of the possible ejections, the bowling ball will slow even more, albeit minimally. However, after enough absorptions (nearly 20,000 for a sodium atom), the atom will near absolute zero. [5] Because a laser can induce up to 10^7 absorptions per second, this process can occur in milliseconds.

With this new technology, scientists could finally examine Einstein's predictions about the BEC. On June 5, 1995, University of Colorado Boulder became the first to synthesize a Bose-Einstein Condensate with Rubidium (Rb) atoms at near 170 nK. [6] That's 1.7×10^{-7} Kelvin above absolute zero, which was the coldest known temperature in the universe! Within a few months, MIT researchers also synthesized a BEC, but with Sodium (Na) atoms. [7] Both teams received the 2001 Nobel Prize for their foray into a new section of physics. [3]

The study of BECs recently underwent even more dynamic and interesting developments. In 1998, just three years after a BEC was first realized, Lene Hau of Harvard managed to slow light down to 17 meters per second by trapping it within a BEC. Within the following years, Hau and her team managed to not only slow light down, but capture it outright. They stopped a pulse of light within a BEC, released it, and then recaptured the light packet within another BEC. This raises the possibility of new developments in light based communications, optical data storage, and quantum computing.

Today, BECs can be made with a variety of atomic species, including Chromium (Cr) [8] and the rare-earth (lanthanide) atoms Erbium (Er) [9] and Dysprosium (Dy) [10], which possess large magnetic dipole moments. Further, BECs are used for a variety of purposes, including the investigation of matter's fundamental low-temperature properties. For example, superfluid flow and quantized circulation (quantum vortices) were directly observed in experiments with BECs. [11] [12] The National Institute of Standards and Technology in Gaithersburg, Maryland carried out many of these experiments. However, the experiments involved atoms that possess negligible dipole moments (like the alkali atoms). Today, there is a strong effort to learn more about the nature of BECs of dipolar atoms, the so-called dipolar BECs. Unlike most atoms, which interact via the short-range Van der Waals force, dipolar atoms interact via the dipole-dipole force, which is anisotropic and long-range. When oriented side-by-side, two dipoles repel each other, and when oriented head-to-tail, they attract. For this reason, experiments with dipolar BECs usually work in nearly two dimensional (2D) geometries, with the dipoles polarized perpendicular to the 2D plane. This results in a configuration where the dipoles are oriented side-by-side, and exert repulsive forces on each other, which helps to stabilize the BEC. The presence of the dipole-dipole force has been shown to result in interesting phenomena in dipolar BECs, and a growing body of theoretical work provides motivation for future experiments. [13]

The attractive interaction results in a unique excitation called a roton excitation. A roton

excitation allows for a particle with high momentum and short wavelength to be excited out of the condensate for very low energy. This is atypical. In order to explore how rotons affect a BEC and its superfluid properties, we first need to understand the mechanisms already in place to describe BECs. From a non-interacting case, to contact interactions, and finally to dipolar interactions, the following section will provide the knowledge to understand the functioning of the BEC.

II.   THEORY

A.   Non-Interacting Bose Gas

The original phenomenon of Bose Einstein condensation focused on a non-interacting gas. Einstein predicted Bose Einstein condensation because it is a purely statistical phenomenon. This means, that a zero interaction BEC exists only due to quantum statistics and not to interactions between molecules, intra molecular interactions, or the nature of the particle. [4] For a BEC to exist, there must be a multitude of particles. A single particle cannot become a BEC. Being a purely statistical phenomenon requires large numbers of particles, despite that each individual particle has no idea any other particles are around it. As mentioned in equation 3, the Bose Distribution governs the statistical occupation of the condensate,

$$n_i = \frac{g_i}{e^{(\epsilon_i - \mu)/k_b T} - 1} \tag{4}$$

Here, ϵ_k denotes the energy of a single particle state, μ is a chemical potential that acts as a Lagrange multiplier to fix particle number in the Grand Canonical Ensemble. In order to achieve this, the condition is applied that the sum of all individual occupancies must sum to the total particle number. k_b is the Boltzmann constant and T is temperature. We fully calculate the Bose-Einstein Distribution because we know the energy levels in free space for non-interacting particles. When interactions appear in our condensate, calculating becomes significantly more difficult, as we will discuss later. Based solely on quantum statistics, the Bose Distribution dictates the occupancy of each state, ground and excited. [4]

At high temperatures, these quantum statistics are meaningless. Any large temperature precludes the ability for a condensate to form and the statistics that govern it become the Maxwell Boltzmann statistics mentioned in Eq 1 [2].

However, Bose-Einstein statistics are required near the critical temperature because quantum statistics fully control the arrangement of bosons at such low temperatures. The critical temperature, defined as the highest temperature at which the macroscopic occupation of the lowest energy state appears, is the lowest temperature in which all atoms are in excited states (no atoms in the ground state). The maximum values of particles in excited states results in a μ of zero. The equation for number of particles in an excited state is given as followed $N_{ex} = \int dk/(2\pi)^3 f_k$. f_k is the same as the Bose-Einstein distribution mentioned earlier in Eq 3, we simply replace n_i with f_k (k being the new momentum quantum number). The integral is then rewritten in terms of $x = \epsilon/k_b T_{crit}$ and evaluated. After changing bases back and solving, the calculations result in [14]

$$T_{crit} = \frac{2\pi\hbar^2}{mk_b} \left[\frac{N}{V\zeta(3/2)} \right]^{2/3} \tag{5}$$

If we rewrite the result of Eq 5, we get $N_{ex} = N(T/T_{crit})^\alpha$. We find $N_0/N = [1-(T/T_{crit})^\alpha]$ where α is 3/2 for particles in a 3 dimensional (3D) box of volume V. [14] This result provides for the graphing of the condensate fraction versus the temperature critical temperature fraction which will result in Fig. 1. The non-interacting BEC is a purely quantum statistical phenomenon able to be efficiently and gracefully described using the above equations.

The Schrodinger equation is the only way to describe a quantum object. This equation $-\hbar^2/2m^2\nabla^2\Psi + U\Psi = E\Psi$ can describe the motion and energy levels of any particle. [4] Schrodinger's equation can be written as $\hat{H}\Psi_i = E\Psi_i$, because the left side of Schrodinger's equation is called the Hamiltonian ($\hat{H}$). $\hbar$ is a constant and m is the mass of the particle. In order to simplify future equations, we set $\hbar = m = 1$. Ψ is the wave equation that describes the motion of the particle. We can solve the differential equation for various Ψ's depending upon the potential energy function, U, used in the equation. For a particle in free space (U=0), we find that the type of waves that satisfy it are plane waves that take the form, $\Psi = Ae^{\imath kx - \imath\omega t}$. Although there are two types of Schrodinger's equations, time independent and time dependent, we use the time independent equation because it gives us energies which we can plug into the Bose-Einstein distribution. Without the time component, plane waves take the form $\Psi = Ae^{\imath kx}$ with $A = 1/L^{D/2}$. D is the dimensionality of our system. L is the length scale over which the quantum object is being described. If confined by an infinite potential well, L is the distance between the two walls of the well. In order to find the
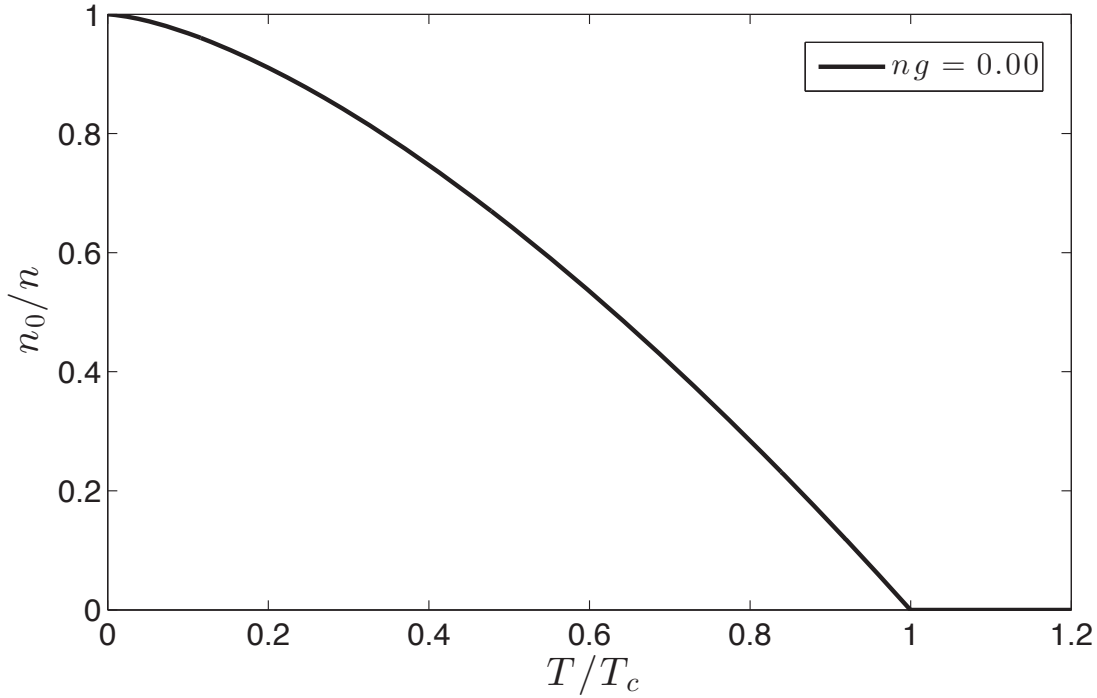
FIG. 1: Condensate fraction of a non-interacting BEC. At 0 Kelvin, every atom is within the condensate. As temperature increases, the condensate fraction begins to drop, more slowly at first, but gradually sharpens as it reaches the critical temperature. Beyond the critical temperature, every atom occupies the thermal gas.

energy levels the particle can inhabit, we operate on the wave equation with the quantum Hamiltonian operator. The return of this operation gives the eigenvalues, or energy levels, of $\epsilon_k = \hbar^2 k^2 / 2m$. k is a quantum number describing the state level that the particle occupies. At low energy in a box of length L^D, the particle can only exist in discrete energy levels. [14] At higher energies with no potential, the energy levels are not defined.

Since non-interacting Bose-Einstein condensation is purely statistical, we require large numbers of particles. For a BEC without interactions, each particle feels no effect from any other particle. Therefore, we treat the condensate and surrounding thermal cloud as a collection of individual particles. For a single particle, the Schrodinger Equation gives the energy which we can plug into the Bose Distribution for a given temperature.

B. The Challenge Posed By Interactions

If we have several interacting atoms to describe, the Hamiltonian becomes more complicated $\hat{H} = \sum_i \frac{\hat{p}^2}{2m_i} + \sum_i U(\vec{r}_i) + \frac{1}{2} \sum_{ij} V(\vec{r}_i - \vec{r}_j)$. In order to account for the interactions between particles, we add a term to our potential to describe particle-particle interactions. An example of what this terms would look like is $V(\vec{r}_i - \vec{r}_j) = \sum_{\vec{r}} \frac{1}{4\pi\varepsilon_0} e^2/|\vec{r}_i - \vec{r}_j|$ in the case of electrons repelling via Coulomb interactions. For one particle, our solution remains unchanged. For two particles, the solution becomes more difficult but still reasonable. When our particles do not interact, our wave equation is a product of single particle wave functions. When our particles do interact, each particle must remain symmetric upon exchange with any other particle. As we increase particle number drastically, keeping track of each particle's interactions with every other one is absurdly difficult.

This difficulty is compounded by the difference between bosons and fermions and how they cause the wave functions to alter. When we have the requirement that particles are indistinguishable like our particles, we must ensure that the probability density of the wave function remains symmetric upon exchange. If the probability density differed when swapping two particles, then the particles are distinguishable. There are two methods of constructing a wave function to describe indistinguishable particles. The first method is by ensuring that when swapping two particle positions, the sign of the wave equation remains the same $\Psi(x_1, x_2) = \Psi(x_2, x_1)$. This is called a symmetric wave function. The second method is by ensuring that when two particles swap positions, a minus sign is introduced $\Psi(x_1, x_2) = -\Psi(x_2, x_1)$. This is called an antisymmetric wave function. When a particle is under symmetric exchange, it is a boson. Likewise, particles which are antisymmetric under exchange are fermions. Since we model bosons, all our particles will be symmetric under exchange.

The Schrodinger equation is capable of fully describing a single particle. With two particles, we treat the system in its center of mass frame (effectively framing it as a one particle problem). As we transition to a many body system, particle number ranging from 3 to infinity, the equations become increasingly difficult to solve. We can numerically solve many body problems, but computations become exponentially more difficult the more particles we encounter. Every particle has 3 spatial degrees of freedom (in 3D) so a system of N particles has 3N degrees of freedoms. In addition, a symmetric wave function requires that the wave

function remain unchanged through particle exchange. This means a system of two particles would look like $\Psi = \frac{1}{\sqrt{2}}(\psi_1(r_1)\psi_2(r_2) + \psi_1(r_2)\psi_2(r_1))$. A system of 3 particles would have 6 terms in it. As we increase the number of particles in the system, the number of terms increases as $N!$. Even if N is just 20, the number of terms in our wave function we need to keep track of is over 2.4×10^{18}. The systems of particles we intend to work with will be much more vast than twenty so a different method of computation is required. We describe our method for overcoming these difficulties, "second quantization", in Section II C.

1. Contact Interactions

Although we will focus on dipolar interactions, most atoms exhibit contact interactions. Van der Waals interactions, the normal mode for most atoms, model contact interactions well. Particles with contact interactions can be thought of as pool balls. Our particles move and if they make contact with another particle, they exchange energy and momentum. The particles then move off in different directions, until they interact with another particle. Without contact interactions, the particles would pass right through each other, completely unaware of the other's existence.

We model contact interactions using a delta function and a contact-interaction coupling, g, which we can adjust to model stronger and weaker interactions. Physically, g corresponds to a ratio of the scattering length of the particles to the inter particle spacing. For our dilute gas, $g = 1$ models the strongest interaction strength we will use while $g = 0$ models no interactions at all. The potential itself manifests as $V(\vec{r} - \vec{r'}) = g\delta(\vec{r} - \vec{r'})$.

2. Dipolar Interactions

Magnetic moments in an atom come from the total accumulation of electron spin added with orbital angular momentum of that specific atom. The magnetic dipole moment can be calculated from the total angular momentum of the atom via the following equation $m_{\text{Atom}} = g_J \mu_B \sqrt{j(j+1)}$. In this equation, j is the total angular momentum and g_J and μ_B are constants.

For dipoles with dipole moments of magnitude d all polarized in the same direction, $\hat{d} = \hat{z}$ describes a dipole moment polarized along the z axis. [15] The dipole dipole interaction

potential is given by

$$V_d(\vec{r}) = d^2 \frac{1 - 3\cos\theta}{|\vec{r} - \vec{r'}|^3}.$$ (6)

.

θ is the angle between the dipole polarization vector $\hat{d}$ and the vector separating the dipoles $|\vec{r} - \vec{r'}|$. [16] Vectors $\vec{r}$ and $\vec{r'}$ give the position of the two dipoles and d is the dipole moment. These can range from zero to completely non-interacting (in terms of dipoles) to upwards of 10 in dimensionless units. This constant is determined by the physical dipole moments of the atom.

The dipole-dipole interaction can be attractive or repulsive. When the dipoles align horizontally, they repel each other. When the dipoles instead align vertically, they attract. This causes problems for a BEC because particles that are too attractive will acquire too much kinetic energy exciting them out of the BEC. In order to avoid this, we confine our particles to a 2D geometry so they only sample the repulsive potential of each other. However, as the momentum of the particles increases, their size nears the same order as the trap. This allows the bosons to begin to sample the attractive potential of dipolar interactions as oppose to solely the repulsive component.

C.   Second Quantization

An alternative method to perform calculations without the Schrodinger is by using 2nd Quantization. Our calculations for 2nd Quantization are shown in appendix A. 2nd Quantization represents the quantum many body states as part of a Fock state basis. Fock states are a form of a Hilbert space. Hilbert space is an infinite orthonormal basis set. For example, Cartesian coordinates scale where each new dimension becomes orthonormal to all previous dimensions. As we gain more dimensions, the orthogonality of our basis remains the same. Fock states are a quantum mechanical states that have well defined number of particles within them. For example a state with 2 particles in the ground state, none in the first excited state, three in the second excited state, and seven in the third excited state will appear as $\chi_1 = |2, 0, 3, 7, ... >$. In order to change the number of particles in a certain state, we operate on it withe a creation or an annihilation operator. $\hat{a}_{k=2}\chi_1 = \sqrt{n}|2, 0, 4, 7, ... >$

where n is the number of states in the wave equation.

In order to convert from our Hilbert space wave function to Fock space wave functions (Fock space is a specific type of Hilbert space), we expand $\hat{\Psi}$ and $\hat{\Psi}^\dagger$ as a function of an orthonormal basis with creation and annihilation operators. [3]

$$\hat{\Psi}^\dagger(\vec{r}) = \frac{1}{\sqrt{L^d}} \sum_{\vec{k}} e^{-i\vec{k}\cdot\vec{r}} \hat{a}_{\vec{k}}^\dagger \tag{7}$$

$$\hat{\Psi}(\vec{r}) = \frac{1}{\sqrt{L^d}} \sum_{\vec{q}} e^{-i\vec{q}\cdot\vec{r}} \hat{a}_{\vec{q}} \tag{8}$$

Creation and annihilation operators do exactly what their names suggest. One creates a particle in a specific state and one annihilates a particle in a specific state. [3] Creation and annihilation operators are referred to as raising and lowering operators as well.

$$\hat{a}_{\vec{k}_i}^\dagger |N_{\vec{k}=0}, N_{\vec{k}_1}, ..., N_{\vec{k}_i}, ... > = \sqrt{N_{\vec{k}_i} + 1} |N_{\vec{k}=0}, N_{\vec{k}_1}, ..., N_{\vec{k}_i} + 1, ... >$$
$$\hat{a}_{\vec{k}_i} |N_{\vec{k}=0}, N_{\vec{k}_1}, ..., N_{\vec{k}_i}, ... > = \sqrt{N_{\vec{k}_i}} |N_{\vec{k}=0}, N_{\vec{k}_1}, ..., N_{\vec{k}_i} - 1, ... > \tag{9}$$

We substitute these new Ψs into our non-interacting Hamiltonian,

$$\hat{H} = \int \Psi^\dagger(\vec{r}) \hat{H}^1 \Psi(\vec{r}) d\vec{r}, \tag{10}$$

and using several commutation relations as seen in Appendix A, arrive at a new Fock space Hamiltonian.

$$\hat{H}^1 = \sum_{\vec{k}} \hat{a}_{\vec{k}}^\dagger \hat{a}_{\vec{k}} \frac{k^2}{2} \tag{11}$$

We use ensure our Hamiltonian maintains symmetry by relying on the commutation relations. Our system is made up entirely of bosons so we require a symmetric wave function, despite introducing second quantization.

In addition, we replace $k^2/2$ with $\epsilon_{\vec{k}}$; the single particle energy term. $\epsilon_{\vec{k}}$ describes the kinetic energy of a single particle. Once summed, we can find the kinetic energy of our whole

system. As previously mentioned, μ is the chemical potential and serves as a Lagrange multiplier to fix particle number. It appears here from writing our Hamiltonian in the grand-canonical ensemble,

$$\hat{H} = \sum_{\vec{k}} \hat{a}_{\vec{k}}^\dagger \hat{a}_{\vec{k}} \left(\epsilon_{\vec{k}} - \mu \right). \tag{12}$$

We perform the same type of operation for the interaction potential and the interaction Hamiltonian.

$$\hat{H}_{int} = \frac{1}{2L^D} \sum_{\vec{k},\vec{p},\vec{q}} \widetilde{V}(\vec{q}) \hat{a}_{\vec{q}-\vec{q}}^\dagger \hat{a}_{\vec{k}+\vec{p}}^\dagger \hat{a}_{\vec{k}} \hat{a}_{\vec{p}} \tag{13}$$

Here, $\widetilde{V}(\vec{q})$ is the Fourier transform of $V(\vec{r})$ into momentum space. This term describes the exchange of momentum between two particles when they interact. We've begun with modeling a general contact interaction potential, but eventually will focus on magnetic dipoles in a quasi-2D case.

D. Mean Field Approximation

Despite the work put into second quantization, our Hamiltonian is still not in a practical form. We need the Hamiltonian to be diagonal and quadratic in order to find its eigenvalues (energy of the system). Our first step in simplifying our equations is to make the mean field approximation The mean field approximation claims since we are going to working with such large numbers of particles in the ground state, we can approximate $\hat{a}_{\vec{k}=0}$ and $\hat{a}_{\vec{k}=0}^\dagger$ as a number, $\sqrt{N_0}$. Normally, $\hat{a}_{\vec{k}=0}$ would return a value of $\sqrt{N_0 - 1}$. [17] The difference between a single particle in a state consisting of 100,000 is insignificant. When our quantum numbers p, q, and k equal zero, all the operators go to $\sqrt{N_0}$. This leads to a factor of N_0^2 appearing in front of the interaction term. $N_0^2 \widetilde{V}(0)/2L^D$ is the coefficient in front of the zeroth order interaction portion of the Hamiltonian. This component of the Hamiltonian models the interactions between the condensate atoms and other condensate atoms. Next we should set three operators to $\hat{a}_0$. When these three operators all go to $\sqrt{N_0}$, we will only have one operator left. However the expectation value of a single operator is always

zero. This stems from the fact that an expectation value needs two equal energy states with the same particle number or the expectation value goes to zero. When a raising or lowering operator acts upon our initial state, particle number will change. Since we will have same momentum states with with different particle number , the expectation value is zero. As a result, $\hat{H}_3$ will also go to zero. Since we need to return Fock states with the same particle number in order to calculate with them, no system of 2 raising and 1 lowering or 2 lowering and 1 raising operator will return our initial particle. $\hat{H}_1 = \hat{H}_3 = 0$. $\hat{H}_2$, which models the interactions between the condensate atoms and the excited gas, will be non-zero. We send two operators from $\hat{a}_0$ to $\sqrt{N_0}$, and keep two to act upon the state. There are six combinations two operator systems when two operators are approximated to $\hat{a}_0$. The first term is when the two creation operators are set to zero, $q = p$ and $k = -p$. This leaves us with $N_0/2L^D \widetilde{V}(\vec{p})\hat{a}_{-\vec{p}}\hat{a}_{\vec{p}}$. The N_0 is due to the two $\hat{a}_0$ we approximated, and the remaining operators are summed over one quantum number. A full derivation can be found in Appendix A. After completing the decomposition for all six possible combinations, we find

$$\hat{H}_{0,2} = \frac{(N_0)^2 \widetilde{V}(0)}{2L^D} + \frac{N_0}{2} \sum_{\vec{k}} \{\widetilde{V}(\vec{k}) \left(\hat{a}_{\vec{k}}^{\dagger}\hat{a}_{-\vec{k}}^{\dagger} + \hat{a}_{\vec{k}}\hat{a}_{-\vec{k}} + 2\hat{a}_{\vec{k}}^{\dagger}\hat{a}_{\vec{k}}\right) + 2\widetilde{V}(0)\hat{a}_{\vec{k}}^{\dagger}\hat{a}_{\vec{k}}\} + \hat{H}_4. \quad (14)$$

.

E.   Hartree-Fock Decomposition

$\hat{H}_4$ which describes the thermal cloud interacting with other thermal cloud atoms. This portion of the Hamiltonian describes our momentum numbers, k, q, and p, when none can equal zero. Only atoms in excited states contribute to $\hat{H}_4$. When we add $\hat{H}_0$, $\hat{H}_2$, and $\hat{H}_4$, we have the Bogoliubov Hamiltonian. This Hamiltonian can describe an interacting condensate and its surrounding gas. The interaction component of the Hamiltonian is now mostly in quadratic operators, but it is not yet diagonal so we cannot compute the eigenvalues for several reasons. In order to be diagonal, the Hamiltonian must have no off-diagonal elements. A series of operators with equal numbers of creation and annihilation operators is diagonal while having an inequality between the number of operators constitutes being off diagonal.

$\hat{H}_4$ remains undiagonalized and quartic in terms of its operators. We cannot use a mean field approximation on $\hat{H}_4$, because none of the operators can go to $\hat{a}_0$ because we've already eliminated these terms. Therefore, we need to use a different method of obtaining an equivalent quadratic form. Hartree-Fock decomposition will achieve exactly this. In Hartree-Fock, you take every possible ordered pair of operators and create expectation values of them, leaving the other two operators unchanged. For our four operator term, this will leave six terms. Once we have our six terms, we set equivalences that will reduce the number of quantum numbers to two. The $\hat{a}_{\vec{k}}^{\dagger}\hat{a}_{\vec{k}}$ operators and our densities, normal and anomalous, stay in terms of the quantum number $\vec{k}$. $\vec{p}$ and $\vec{k}$ are allowed to appear without reservation in the other portions of the term. We define our normal density to be the expectation value of $\widetilde{n}(\vec{k}) = \hat{a}_{\vec{k}}^{\dagger}\hat{a}_{\vec{k}}$, because it results in a number operator and when combined with $1/L^D$, will provide the actual density for our condensate. [17] We also define an anomalous density to be $\widetilde{m}(\vec{k}) = \hat{a}_{\vec{k}}\hat{a}_{\vec{k}}$. After applying these definitions, our $\hat{H}_4$ turns into

$$\hat{H}_4 \simeq \widetilde{n}\widetilde{V}(0) \sum_{\vec{k}\neq 0} \hat{a}_{\vec{k}}^{\dagger}\hat{a}_{\vec{k}} + \frac{1}{2L^D} \sum_{\vec{k}\vec{q}\neq 0} \widetilde{V}(\vec{q}-\vec{k}) \left(\widetilde{n}(q)\hat{a}_{\vec{k}}^{\dagger}\hat{a}_{\vec{k}} + \widetilde{m}(q)\hat{a}_{\vec{k}}^{\dagger}\hat{a}_{-k} + h.c. \right) \qquad (15)$$

We now redefine our quadratic (but off diagonal Hamiltonian) to be $H_2' = H_2 + H_4$. This equation can be made much simpler by substituting the self energies $\Sigma_{11}(\vec{k})$, $\Sigma_{20}(\vec{k})$, and $\Sigma_{02}^*(\vec{k})$. [18] We set $\Sigma_{11}(\vec{k})$ equal to all components of the already diagonalized portion. We set $\Sigma_{20}(\vec{k})$ and its Hermitian $\Sigma_{02}^*(\vec{k})$ equal to all the components in front of the off diagonal operators. This simplifies our equation and produces the result below

$$\hat{H}_2' = \sum_{\vec{k}\neq 0} \left(\epsilon_{\vec{k}} - \mu + \Sigma_{11}(\vec{k}) \right) \hat{a}_{\vec{k}}^{\dagger}\hat{a}_{\vec{k}} + \frac{1}{2} \sum_{\vec{k}\neq 0} \left(\Sigma_{20}(\vec{k})\hat{a}_{\vec{k}}\hat{a}_{-k} + \Sigma_{02}^*(\vec{k}\hat{a}_{\vec{k}}^{\dagger}\hat{a}_{\vec{k}}^{\dagger}) \right) \qquad (16)$$

Where

$$\Sigma_{11}(\vec{k}) = n\widetilde{V}(0) + n_0\widetilde{V}(\vec{k}) + \frac{1}{L^D} \sum_{\vec{q}\neq 0} \widetilde{n}(\vec{q})\widetilde{V}(\vec{q}-\vec{k}) \qquad (17)$$

$$\Sigma_{20}(\vec{k}) = n_0\widetilde{V}(\vec{k}) + \frac{1}{L^D} \sum_{\vec{q}\neq 0} \widetilde{m}(\vec{q})\widetilde{V}(\vec{q}-\vec{k}) \qquad (18)$$

If we did not want to include $\hat{H}_4$ in our calculations, we could obtain the results from only $\hat{H}_2$ by setting our densities, $\widetilde{n}(\vec{q})$ and $\widetilde{m}(\vec{q})$, to zero and moving n to n_0. This result

would account for interactions between the condensate and itself and the condensate and the thermal gas surrounding the condensate (but no interactions between the gas and other thermal atoms).

F.   Bogoliubov Transformation

Next we diagonalize H_2' in order to extract its eigenvalues. Diagonalized matrices are necessary for calculations, because we need to find the eigenstates of the Hamiltonians to find the energies of the system to plug into the Bose-Einstein distribution (Eq 3). [19] An off diagonal Hamiltonian has off center elements such as quadratic operators which feature two creation operators. Diagonal elements need equal numbers of creation and annihilation operators. We use the Bogoliubov transformation in order to achieve this. The Bogoliubov transformation uses a change of variable to change any non-diagonal quadratic equation into a diagonal one. [15] Unfortunately, the transformation does not conserve particle number so we introduce the grand-canonical ensemble and μ to conserve particle number. Bogoliubov originally invented this method in order to solve mathematical difficulties with superconductivity. In order to do this, Bogoliubov redefined the creation and annihilation operators as follows

$$\hat{a}_{\vec{k}} = u_{\vec{k}}\hat{b}_{\vec{k}} + v_{\vec{k}}^*\hat{b}_{-\vec{k}}^\dagger \tag{19}$$

$$\hat{a}_{\vec{k}}^\dagger = u_{\vec{k}}^*\hat{b}_{\vec{k}}^\dagger + v_{\vec{k}}\hat{b}_{-\vec{k}} \tag{20}$$

Instead of our previous operators that create or destroy single particles, our new operators $\hat{b}_{\vec{k}}$ and $\hat{b}_{\vec{k}}^\dagger$ create and annihilate quasi-particles. Quasi-particles are a mathematical tool used to describe complicated systems. Instead of modeling a difficult system of many moving particles strongly interacting with each other, we can model quasi-particles which act more like non-interacting particles. Keep in mind quasi-particles are not "real" particles and instead a mathematical representation used to simplify the system.

To ensure our operators commute, we calculate the commutation relation we previously defined for $\hat{a}_{\vec{k}}$ and $\hat{a}_{\vec{k}}^\dagger$. As opposed to a simple Dirac delta, we find the result $u_{\vec{k}}u_{\vec{k}}^* - v_{\vec{k}}^*v_{\vec{k}} = 1$. First, we plug in our definitions for $\hat{a}_{\vec{k}}$ and $\hat{a}_{\vec{k}}^\dagger$ into the Bogoliubov Hamiltonian, $\hat{H}_2'$. Next, we bundle similar terms; We place all $\hat{b}_{\vec{k}}\hat{b}_{\vec{k}}$, $\hat{b}_{\vec{k}}^\dagger\hat{b}_{\vec{k}}$, and $\hat{b}_{\vec{k}}^\dagger\hat{b}_{\vec{k}}^\dagger$ into separate groups. We find that the

bundle of $\hat{b}_{\vec{k}}\hat{b}_{\vec{k}}$ and $\hat{b}_{\vec{k}}^{\dagger}\hat{b}_{\vec{k}}^{\dagger}$ are conjugates of each other. $(v^2 + u^2 + 2uv)\widetilde{V}(\vec{q}) + 2\widetilde{V}(0)vu + \epsilon_{\vec{k}} - \mu$ lies in front of $\hat{b}_{\vec{k}}\hat{b}_{\vec{k}}$. We have the freedom to choose u and v such that these off diagonal elements vanish. We linearly combine our $\hat{b}_{\vec{k}}\hat{b}_{\vec{k}}$'s coefficients with the boson commutation relation via Mathematica in order to solve for u and v. [4]

$$u_{\vec{k}} = \sqrt{\frac{1}{2}\left(\frac{\epsilon_{\vec{k}} - \mu + \Sigma_{11}(\vec{k})}{\omega_{\vec{k}}} + 1\right)} \tag{21}$$

$$v_{\vec{k}} = -\sqrt{\frac{1}{2}\left(\frac{\epsilon_{\vec{k}} - \mu + \Sigma_{11}(\vec{k})}{\omega_{\vec{k}}} - 1\right)} \tag{22}$$

where

$$\omega_{\vec{k}} = \sqrt{\left(\epsilon_{\vec{k}} - \mu + \Sigma_{11}(\vec{k})\right)^2 - (\Sigma_{20}(\vec{k}))^2} \tag{23}$$

We redefine our normal and anamolous energies using our u and v.

$$\widetilde{n}(\vec{k}) = \left(u_{\vec{k}}^2 + v_{\vec{k}}^2\right)f_{\vec{k}} + v_{\vec{k}}^2 \tag{24}$$

$$\widetilde{m}(\vec{k}) = u_{\vec{k}}v_{\vec{k}}\left(2f_{\vec{k}} + 1\right) \tag{25}$$

After plugging our values for u and v into the our Hamiltonian, simplification brings us the full diagonalized Hamiltonian. Despite the complexity of our system, $\hat{H}_2'$ ultimately arrives at a simple and elegant final form.

$$\hat{H}_2' = E_0' + \sum_{\vec{k}\neq 0}\omega_{\vec{k}}\hat{b}_{\vec{k}}^{\dagger}\hat{b}_{\vec{k}} \tag{26}$$

G.   The Hugenholtz-Pines Theorem

We have yet to find an expression for the chemical potential, μ. We can find the chemical potential by introducing the mean field expansion into the equation of motion for an operator, $\hat{a}_{\vec{k}}$

$$i\hbar\partial_t \hat{a}_{\vec{k}} = \left[\hat{a}_{\vec{k}}, \hat{H}\right]$$

$$= \left(\epsilon_{\vec{k}} - \mu\right)\hat{a}_{\vec{k}} + \frac{1}{L^D}\sum_{\vec{q},\vec{p}}\widetilde{V}(\vec{q})\hat{a}_{\vec{p}}^{\dagger}\hat{a}_{\vec{p}-\vec{q}}\hat{a}_{\vec{k}} \tag{27}$$

$$\mu = n\widetilde{V}(0) + \frac{1}{L^D}\sum_{\vec{k}}\widetilde{V}(\vec{k})\left(\tilde{n}(\vec{k}) + \tilde{m}(\vec{k})\right) \tag{28}$$

In 1959, by a rigorous and general argument, Hugenholtz and Pines determined that μ, the chemical potential, must equal the difference between the two self energies describing the bosonic system; [20]

$$\mu = \Sigma_{11}(0) - \Sigma_{20}(0) \tag{29}$$

$\omega_{\vec{k}}$ is the quasi-particle spectrum that describes the dispersion of non-condensate atoms. At vanishingly small k, this spectrum must produce a gapless Goldstone mode, meaning $\omega_0 = 0$. This is due to the continuous symmetry of the Hamiltonian that is broken by the presence of a condensate. One can easily check that for $\omega_{\vec{k}}$ to be gapless at $k = 0$, the above equation must be satisfied.

H. The Popov Approximation

We can check that when we set $k = 0$ in $\Sigma_{11}(0)$ and $\Sigma_{20}(0)$, we get

$$\Sigma_{11}(0) = n\widetilde{V}(0) + n_0\widetilde{V}(0) + \frac{1}{L^D}\sum_{\vec{q}\neq 0}\tilde{n}(\vec{q})\widetilde{V}(\vec{q}) \tag{30}$$

$$\Sigma_{20}(0) = n_0\widetilde{V}(0) + \frac{1}{L^D}\sum_{\vec{q}\neq 0}\tilde{m}(\vec{q})\widetilde{V}(\vec{q}) \tag{31}$$

We substitute the values gained from Eq. 30 into the Hugenholtz-Pines theorem (Eq. 29), we arrive at the result

$$\Sigma_{11}(0) - \Sigma_{20}(0) = n\widetilde{V}(0) + \frac{1}{L^D}\sum_{\vec{k}}\widetilde{V}(\vec{k})\left(\tilde{n}(\vec{k}) - \tilde{m}(\vec{k})\right)$$

$$\neq \mu. \tag{32}$$

We remember our value for μ in Eq. 28. Unfortunately, our equations don't immediately satisfy the Hugenholtz-Pines theorem. This is a problem, because it indicates our method is somehow flawed. However, we can implement the common Popov approximation in order to comply with Hugenholtz-Pines. The Popov approximation amounts to neglecting the anomalous density, setting $\tilde{m}(\vec{k}) \to 0$. [4, 17]

Physically, $\tilde{m}$ corresponds to two non-BEC atoms simultaneously scattering out of the BEC or scattering into it. This interaction happens, but is not common enough to significantly alter our model of the BEC.

I.   Two Dimensions: the Quasi-Condensate

As we saw in section II B 2, when we consider a gas of polarized dipolar bosons, their interactions are anisotropic. When the dipoles are polarized along the z-axis, the interactions are repulsive in the x-y plane, but are attractive in the z-direction, so two dipoles will attract if aligned in a "head-to-tail" configuration. For this reason it is advantageous to confine the dipolar atoms in the x-y plane with some sort of strong trapping potential, like a harmonic trap in the z-direction, to discourage the gas from collapsing onto itself. Experimentally, this trapping potential can be created using, for example, strong laser beams that attract the atoms into a plane. This results in an effectively two dimensional system where all of the atoms occupy a wave function with a common z-dependence given by $\chi(z)$, which we take to be a Gaussian (the single particle ground state of a 1D harmonic potential). We discuss this further, and derive the reduced contact and dipolar interactions that are relevant for a quasi-2D geometry in Appendix E.

When Bose gas is forced into 2D, it turns out that a true BEC, where every atom occupies the exact same quantum mechanical ground state, can only exist at exactly zero temperature. Thus, no BEC can exist in 2D at any finite temperature [21]. Instead, however, a different low temperature phase can emerge, called a "quasi-condensate," which looks like a BEC on small length scales, but has large thermal fluctuations in the phase of the condensate wave function on longer length scales. This fact was proven rigorously by Mermin and Wagner, and is thus called the Mermin-Wagner theorem. [22] It states that in 2D, continuous symmetries cannot be spontaneously broken at finite temperature in systems with sufficiently short-range interactions (which includes both contact and dipolar $\propto 1/r^3$ interactions). In this

case, the continuous symmetry is the phase symmetry; in our Hamiltonian, the operators can be modified by any phase $e^{i\phi}$ and the Hamiltonian remains unchanged. Essentially, any long-range interaction can induce fluctuations with minimal energy costs, hence altering the ground state of the system.

Heuristically, we find that the Hartree Fock Bogoliubov (HFB) equations can not be solved at finite temperatures in 2D, due to the divergence of the integrals over $\tilde{n}(\vec{k})$. This is a backwards proof of the Mermin-Wagner theorem. So, we need to introduce a new mean-field for the ground state phase. Instead of a pure BEC, we will have a quasi-condensate. The BEC was described by the $k = 0$ (spatially uniform) mode $\sqrt{n_0}$. To determine an analogous mode for the quasi-condensate, we consider the number-phase representation of the Bose annihilation and creation operators that we started with. We can write

$$\hat{\Psi}(\vec{r}) = \sqrt{\hat{n}(\vec{r})}e^{i\hat{\theta}(\vec{r})}. \tag{33}$$

where $\hat{n}(\vec{r})$ is the number operator and $\hat{\theta}(\vec{r})$ is the phase operator. Anticipating there being fluctuations in the phase in a quasi-condensate, we will leave the phase operator as-is, but introduce a mean-field (just a number) for the mean value of the density,

$$\hat{\Psi}(\vec{r}) = \sqrt{n_0 + \delta\hat{n}(\vec{r})}e^{i\hat{\theta}(\vec{r})}. \tag{34}$$

Expanding this to linear order in the operators, which we have to do to match up with our theory above, we find

$$\hat{\Psi}(\vec{r}) = \sqrt{n_0} + i\sqrt{n_0}\hat{\theta}(\vec{r}) + \frac{1}{2}\frac{\delta\hat{n}(\vec{r})}{\sqrt{n_0}}$$
$$\hat{\Psi}^\dagger(\vec{r}) = \sqrt{n_0} - i\sqrt{n_0}\hat{\theta}(\vec{r}) + \frac{1}{2}\frac{\delta\hat{n}(\vec{r})}{\sqrt{n_0}}. \tag{35}$$

We now wish to make a connection with the Bogoliubov theory outlined above in section II F; we move to momentum space, and proceed to represent these operators in momentum space. Enforcing the commutation relation for number and phase operators,

$$\left[\delta\hat{n}(\vec{r'}), \hat{\theta}(\vec{r})\right] = i\delta(\vec{r'} - \vec{r}), \tag{36}$$

we find the momentum space forms to be

$$\delta \hat{n}_{\vec{k}} = \sqrt{n_0}(u_{\vec{k}} + v_{\vec{k}}) \left(\hat{b}_{\vec{k}} + \hat{b}_{\vec{k}}^\dagger \right)$$

$$\hat{\theta}_{\vec{k}} = -\frac{i}{2\sqrt{n0}}(u_{\vec{k}} - v_{\vec{k}}) \left(\hat{b}_{\vec{k}} - \hat{b}_{\vec{k}}^\dagger \right). \tag{37}$$

The normal density of non-condensate atoms can be written as

$$\tilde{n}(k) = n_0 \langle \hat{\theta}_{\vec{k}} \hat{\theta}_{\vec{k}} \rangle + \frac{1}{4n_0} \langle \delta \hat{n}_{\vec{k}} \delta \hat{n}_{\vec{k}} \rangle + \frac{i}{2} \left\langle \left[\delta \hat{n}_{\vec{k}}, \hat{\theta}_{\vec{k}} \right] \right\rangle$$

$$= n_0 \langle \hat{\theta}_{\vec{k}} \hat{\theta}_{\vec{k}} \rangle + \frac{1}{4n_0} \langle \delta \hat{n}_{\vec{k}} \delta \hat{n}_{\vec{k}} \rangle - \frac{1}{2}. \tag{38}$$

Let's look at the behavior of this function at small values of k. We find that it goes like $\tilde{n}(k) \sim T/k^2$ at small k. In 3D, this is convergent at small k when integrating over $\tilde{n}(k)$. In 2D, however, this is divergent! We notice that it's the $n_0 \langle \hat{\theta}_{\vec{k}} \hat{\theta}_{\vec{k}} \rangle$ term in particular that behaves this way. This makes sense, because this term involves the phase operator, and we expect that phase fluctuations to be large and kill the BEC in 2D. Thus, we define the quasi-condensate as

$$n_0 \rightarrow n_0 + n_0 \sum_{\vec{k}} \langle \hat{\theta}_{\vec{k}} \hat{\theta}_{\vec{k}} \rangle. \tag{39}$$

Thus, the quasi-condensate absorbs the phase fluctuations that kill the BEC. The quasi-condensate still has highly suppressed density fluctuations, however.

To fix the HFB equations, we can simply take every term involving $n_0 \langle \hat{\theta}_{\vec{k}} \hat{\theta}_{\vec{k}} \rangle$ and set it to zero. This gives convergent integrals, and a well behaved theory for a quasi-condensate that is perfectly analogous to the 3D theory for the true BEC.

J. Superfluidity

One of the most interesting phenomenon associated with BECs is the emergence of superfluidity. Superfluids are fluids which experience non-viscous flow. are interesting from a practical standpoint because easy access to superfluids would eliminate major losses of

energy and work for industrial applications. Heuristically, it has been found that the super-fluidity coincides with the existence of certain types of BECs. However, not every BECs is a superfluid and not all superfluids stem from the existence of a BEC.

The properties of superfluid BECs were being studied even before the connection between BECs and superfluids was known. Lev Landau, 1962 Nobel Prize winner, analyzed Helium 4, a boson, for its superfluid properties. He showed that Helium 4 would not produce sound wave excitations while flowing past a wall, granted the flow of the Helium was slower than the sound velocity. Landau suggested the sound velocity was also the critical velocity for superfluidity. Above this velocity, Helium 4 is not a superfluid. Unknown at the time, this phenomenon was due to the existence of a BEC within Helium 4. [23]

In order for a BEC to support superfluidity, there needs to be interactions between its constituents. [22] A non-interacting BEC is not a superfluid. Different types of interactions, whether contact or dipolar, could therefore produce superfluidity, and the superfluid fraction (the density of particles acting as a superfluid over the total density of particles) will vary depending on it. Not only do interactions factor into the superfluid (SF) fraction, but the degrees of freedom influence it as well. We calculate the superfluid fraction using the following equation

$$n_s = n_{tot} + \int \frac{d\vec{p}}{(2\pi)^3} \frac{p^2}{3} \frac{\partial}{\partial \omega_{\vec{p}}} f_{\vec{p}}. \tag{40}$$

III. RESULTS

A. Three Dimensions with Contact Interactions

As explained previously, a BEC with no interactions will correspondingly have no super-fluid properties. Therefore, the non-interacting case in three dimensions shown in Fig. 1 from the Non-Interacting Bose Gas Section shows only the condensate fraction. It has no superfluid properties so the superfluid fraction remains at zero.

Contact Interactions bring significantly more interesting results than the non-interacting case. From seeing both the condensate fraction and superfluid fraction on a single plot, we see that contact interactions, no matter how minimal, will give the SF fraction a slight increase over the BEC fraction. As we increase the interaction strength, we similarly find
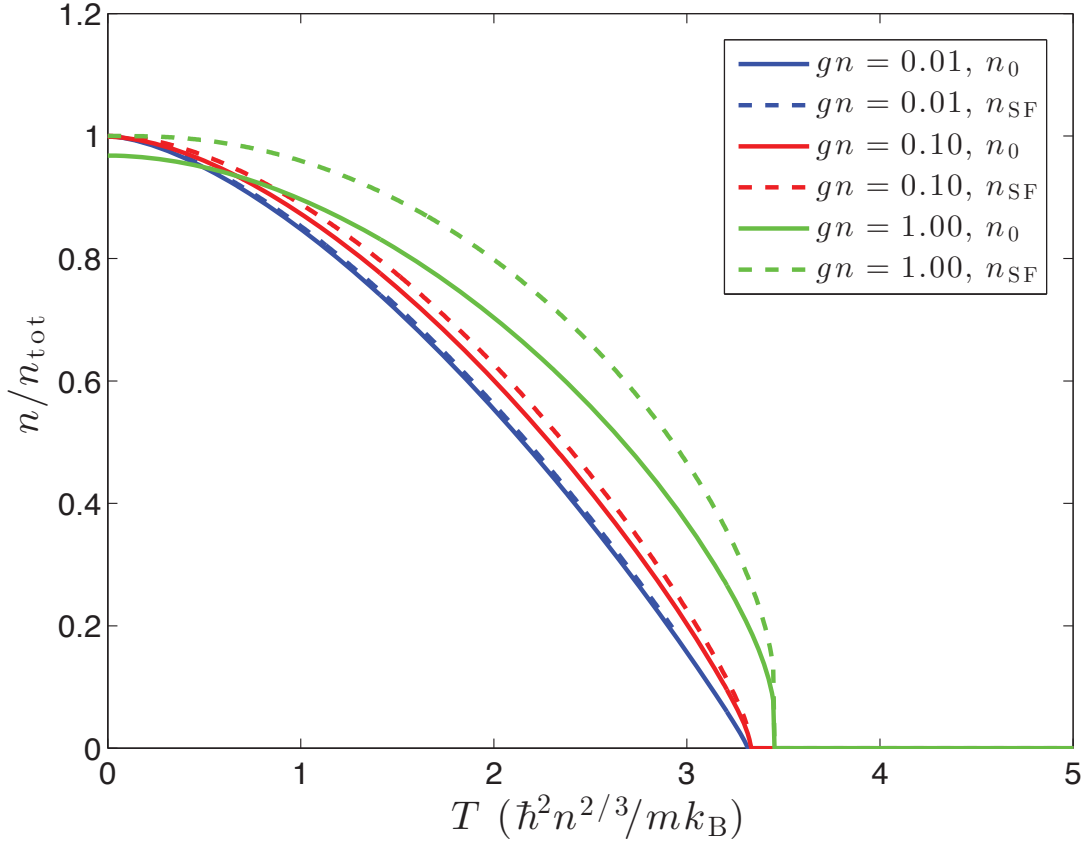
FIG. 2: Condensate fraction and superfluid fraction of a 3 Dimensional BEC with contact interactions of increasing strength. As the contact interaction strength increases, so does the BEC fraction, superfluid fraction, and critical temperature. The superfluid and BEC have the same critical temperature. As temperature nears zero, BECs with very strong contact interactions experience the phenomenon known as quantum depletion in which some atoms occupy the thermal cloud even at zero 0 Kelvin.

a greater increase in the relative increase of the SF fraction over the BEC fraction. We also increasing the contact interaction strength increases the critical temperature. In 3D with contact interactions, the superfluid and condensate crtical temperatures are the same. Near absolute zero, we see the effect of quantum depletion diminishing the condensate. Quantum depletion occurs because of the large strength of the interactions, particles with non-zero momenta are pushed out of the condensate, even at zero temperature. The system is stabilized with excited atoms at zero temperature. On the other end of the temperature scale,
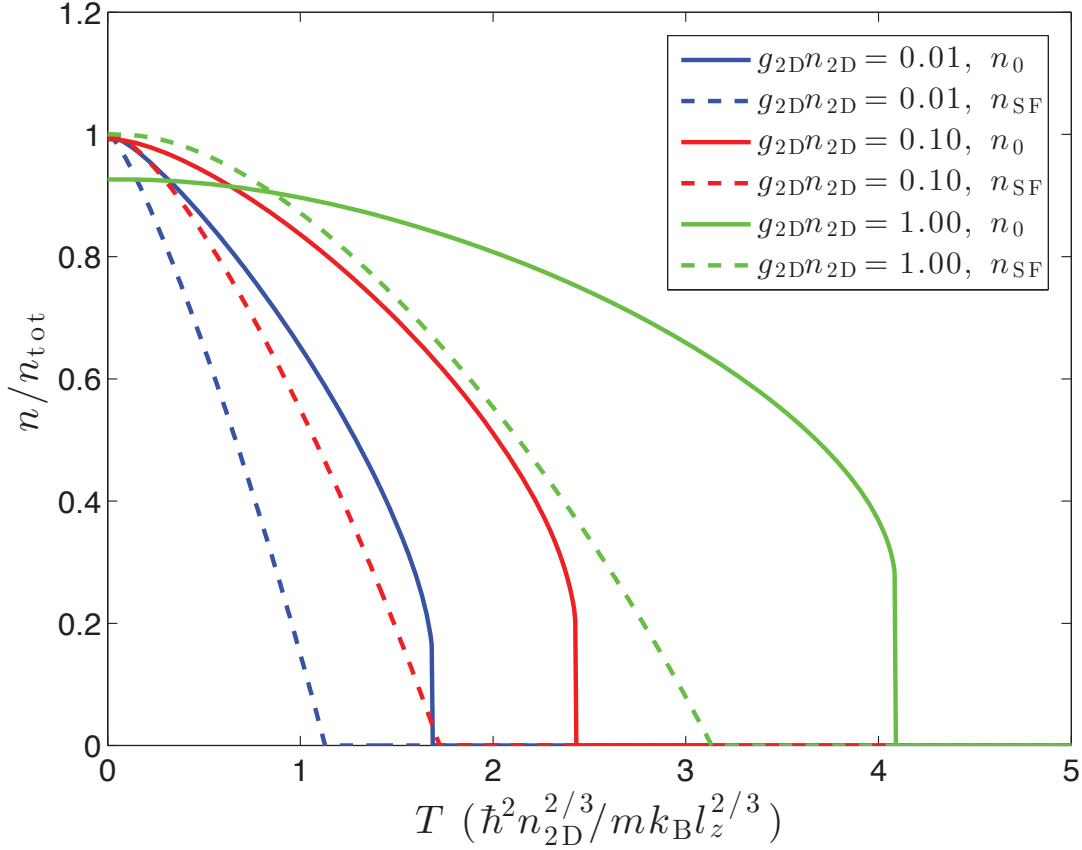
FIG. 3: Condensate fraction and superfluid fraction in a nearly 2D quasi-condensate at finite temperatures with varying strengths of contact interactions. As contact interaction strength increases, so do both the superfluid and condensate critical temperatures: the superfluid critical temperature remains the lower of the two. The occupancy of the condensate and superfluid also increase dramatically with increasing contact interaction strength.

we find that the critical temperature is extended significantly beyond the non-interacting critical temperature. In fact, every increase in contact interaction strength will induce a slightly larger critical temperature.

B. Two Dimensions with Contact Interactions

In 2D, a pure BEC only exists at 0 Kelvin. However, the quasi-condensate functions similar to a pure condensate. The quasi-condensate has supressed density fluctuations and
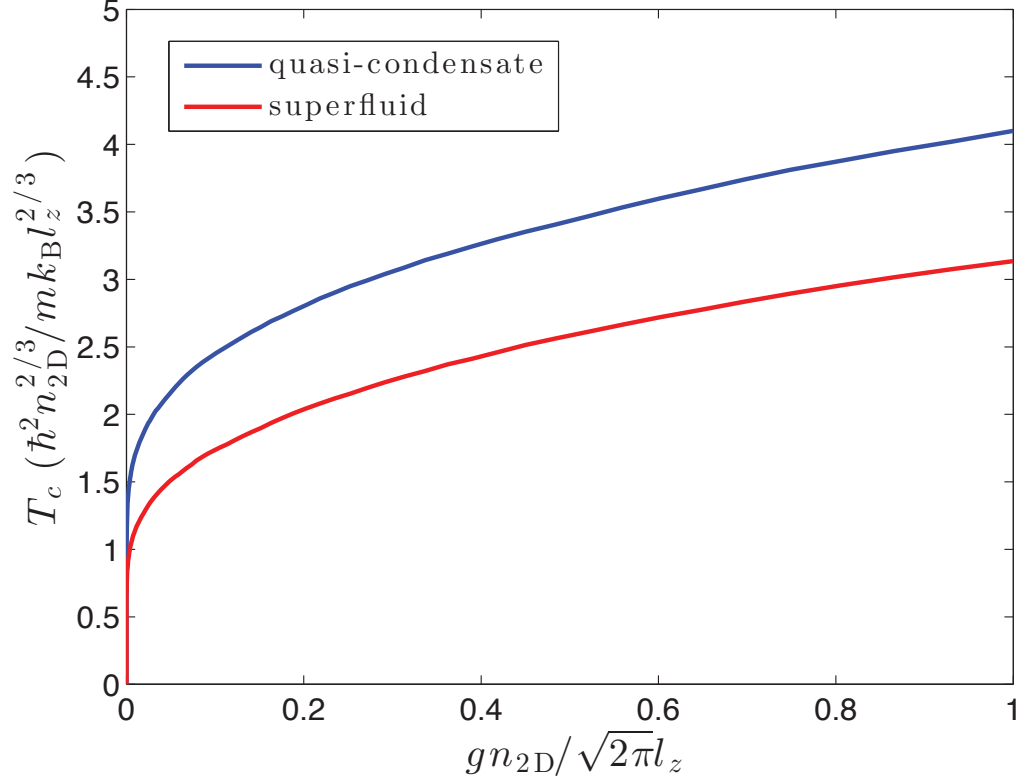
FIG. 4: Critical temperature of the superfluid fraction and the quasi-condensate fraction at different contact interaction strengths while in two dimensions. The critical temperature for the superfluid and the quasi-condensate both initially sharply increase and then continue to increase more slowly with increasing contact interaction strength. However, the superfluid critical temperature spectrum remains moderately lower than the quasi-condensate critical temperature spectrum.

rather large phase fluctuations. Mathematically, these phase fluctuations manifest in the term $\int \frac{d\vec{k}}{(2\pi)^2} u_{\vec{k}} v_{\vec{k}} (2 f_{\vec{k}} + 1)$. This behaves like $\frac{T}{k^2}$ when taken at small k. While this is easily handled in three dimensions, this term diverges in 2D. However, we absorb it into n_0, the condensate density. This converts our condensate from strictly a BEC to a quasi-condensate with phase fluctuations. When taken from a small distance away, a BEC and quasi-condensate act the same. However, at large length scales, a quasi-condensate will change phase whereas a BEC has a single phase. For the purposes of superfluidity and condensate fraction, two dimensions have significant effects.

Similar to 3D, increasing contact interaction strength will likewise increase the condensate fraction. The only temperature where lower interaction strength leads to a higher condensate fraction is near absolute zero when strong interactions cause quantum depletion. However, superfluid fraction is only increased with greater interaction strengths. The increase in contact interaction strength plays a significantly greater role in the increase of the condensate and SF fraction than in 3D. Unlike in 3D, the superfluid has a different critical temperature than the condensate. The condensate undergoes a superfluid transition initially and enters a phase where the condensate still exists, but does not possess superfluid properties. Then the condensate undergoes a transition at its critical temperature into a thermal gas. The two transitions can be seen explicitly in Fig. 4.

A derivation of the quasi-2D contact interaction potentials is given in Appendix D.

C. Two Dimensions with Dipolar Interactiions

The dipolar interaction affects the condensate significantly more dramatically than contact interactions. Although weakly interacting dipoles provide similar condensate fractions and excitation spectrum as those of contact interactions, strong dipolar interactions cause large changes in the condensate's nature. Similar to 2D contact interactions, an increase in the dipolar interaction strength produces an increase in the condensate critical temperature. However, strong dipolar interactions, where rotons are present, such as the green line in Fig. 6 show an enormous quantum depletion. For a moderately strong interaction, $\frac{1}{5}$ of the total particles are in the thermal cloud at zero temperature! Weak Dipolar interactions, where rotons do not manifest, produce higher condensate fractions over the majority of the temperature scale.

Superfluids also undergo an interesting transition. While weakly interacting dipoles produce an increase in superfluid fraction and superfluid critical temperature, the development of the roton excitation spells disater for superfluidity. Not only does superfluidity decrease dramatically, but the roton excitation causes the lower superfluid fraction than more weakly interacting dipoles. In fact, the spectrum of superfluid fraction for strong dipoles concaves. Strong dipolar interactions and roton excitations lead to a dramatic decrease in the condensate's ability to support superfluidity.

Rotons have a profound effect on quantum depletion. As the condensate experiences a
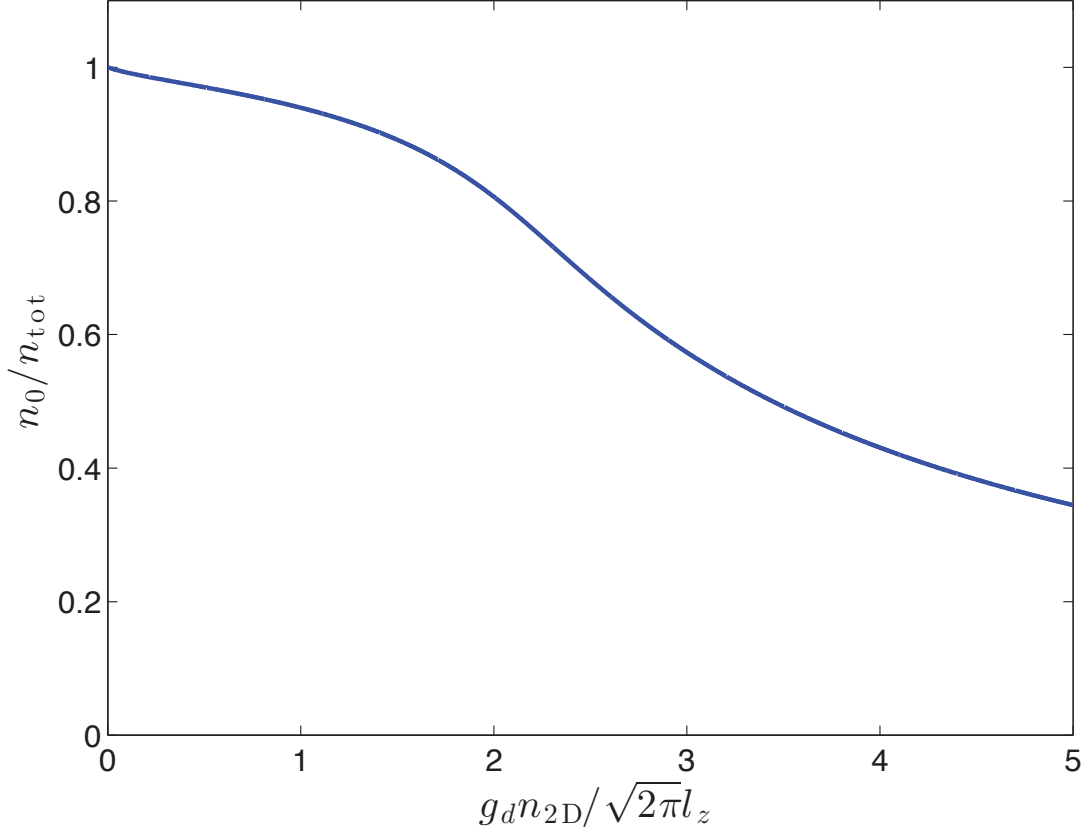
FIG. 5: Effect of dipolar interaction strength on condensate occupancy at 0 Kelvin. The condensate experiences quantum depletion at a consistent rate until dipolar interactions become strong, at which point the roton mode appears and cause a rapid depletion in the condensate.

gradual increase in dipolar interaction strength, a slight linear drop in condensate fraction occurs. Near the dipolar interaction strength of $g_d n_{2D}/\sqrt{2\pi}l_z = 1.5$, the roton mode begins to form and creates a sharp decrease in the condensate occupation at 0 Kelvin. This effect is clearly illustrated in Fig. 5. This steep drop continues until near $g_d n_{2D}/\sqrt{2\pi}l_z = 3$ at which point the condensate fraction at 0 Kelvin begins to return to smaller decreases with increasing interaction strength. We can see the effect of dipolar interaction strength in regards to excitation energy out in Fig. 7. This probably occurs because a roton mode forms near $g_d n_{2D}/\sqrt{2\pi}l_z = 1.5$ which causes lots of low energy excitations. This process compounds with increasing dipolar interaction strength and continues to shrink the condensate fraction at zero temperature.

The spectrum for strong dipolar excitations is unique because with increasing momentum, we find a local maximum (a small hill), and then a wide minimum where the roton excitation exists. If we have only weakly interacting dipoles, then the excitation spectrum looks similar to those of contact interactions. Fig. 8 shows that only at low temperatures do roton modes exist. As temperature increases, many atoms become excited out of the condensate, stabilizing it. If we calculated without a self consistent solution, this would not occur and
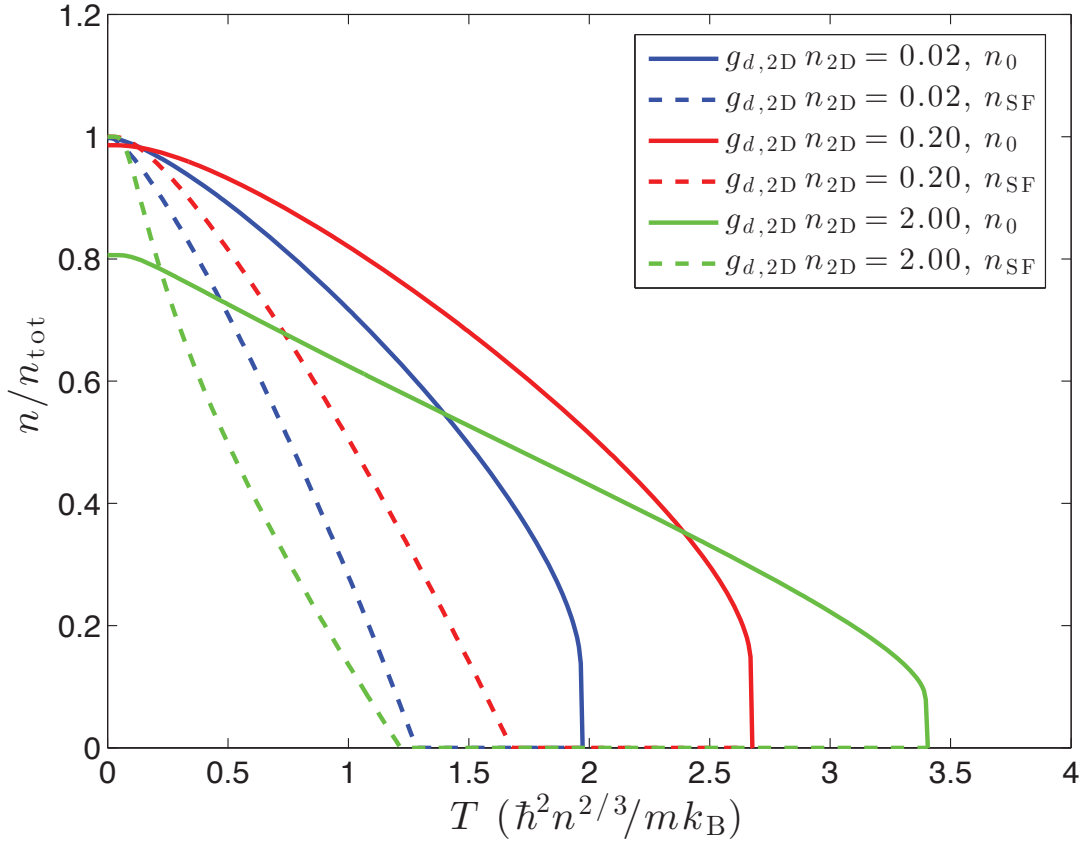

FIG. 6: Condensate fraction and superfluid fraction of a 2 Dimensional BEC with dipolar interactions of varying strength. Increasing dipolar interaction strength vastly increases the critical temperature of the condensate. However, it also drastically increases the effect of quantum depletion, causing condensates with weak dipolar interactions to have a greater condensate fraction at most temperatures. Increasing dipolar interaction strength initially has a positive influence on the superfluid critical temperature and superfluid fraction, but too large of dipolar interactions decreases the critical temperature and superfluid fraction.
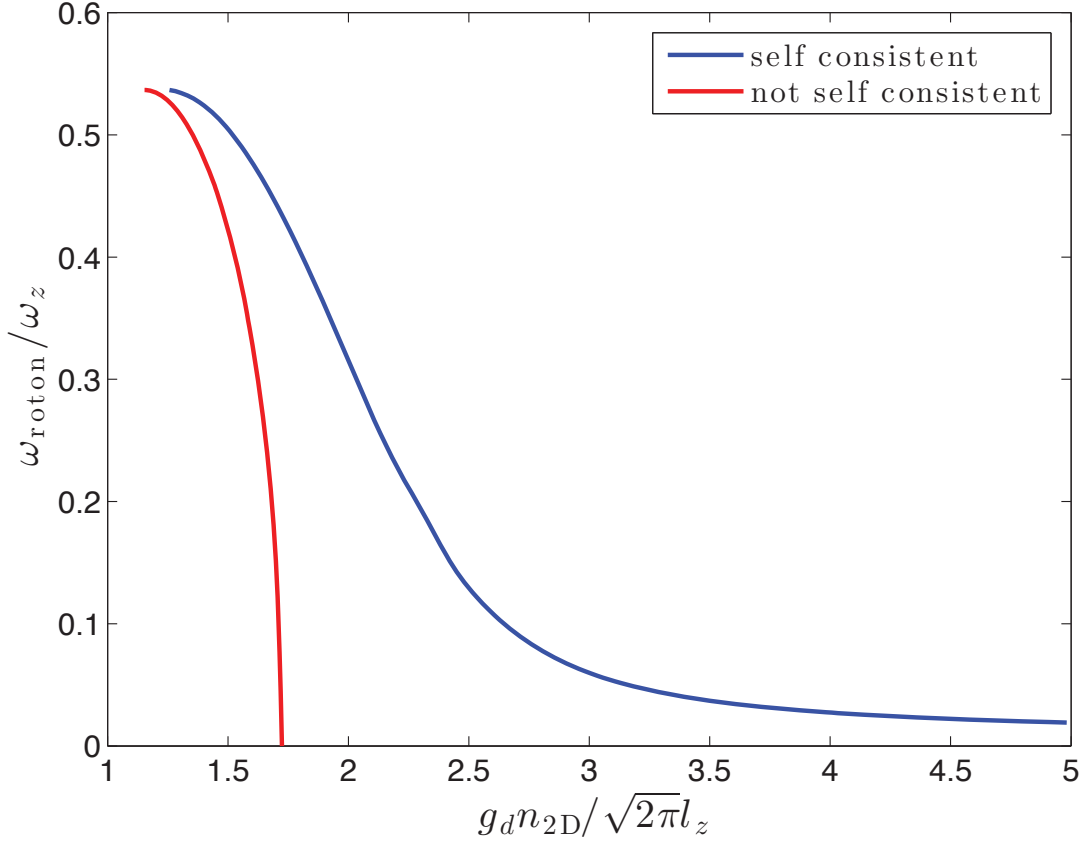
FIG. 7: Comparison of self consistent versus non-self consistent method of calculating in regards to excitation energy of increasing dipolar interaction strength at 0 K. When calculating not self consistently, the roton excitation energy drops to zero soon after roton excitations appear. Therefore, the condensate collapses and all atoms spill into the thermal cloud. However, a self consistent calculation shows that regardless of the interaction strength, the roton excitation energy will never go to zero. As a result, the system is mechanically stable.

instead the condensate would collapse due to the strong dipolar interactions, regardless of temperature. However, we find that temperature acts as a stabilizing mechanism for the condensate, removing the roton mode.

A derivation of the quasi-2D dipolar interaction potentials is given in Appendix E
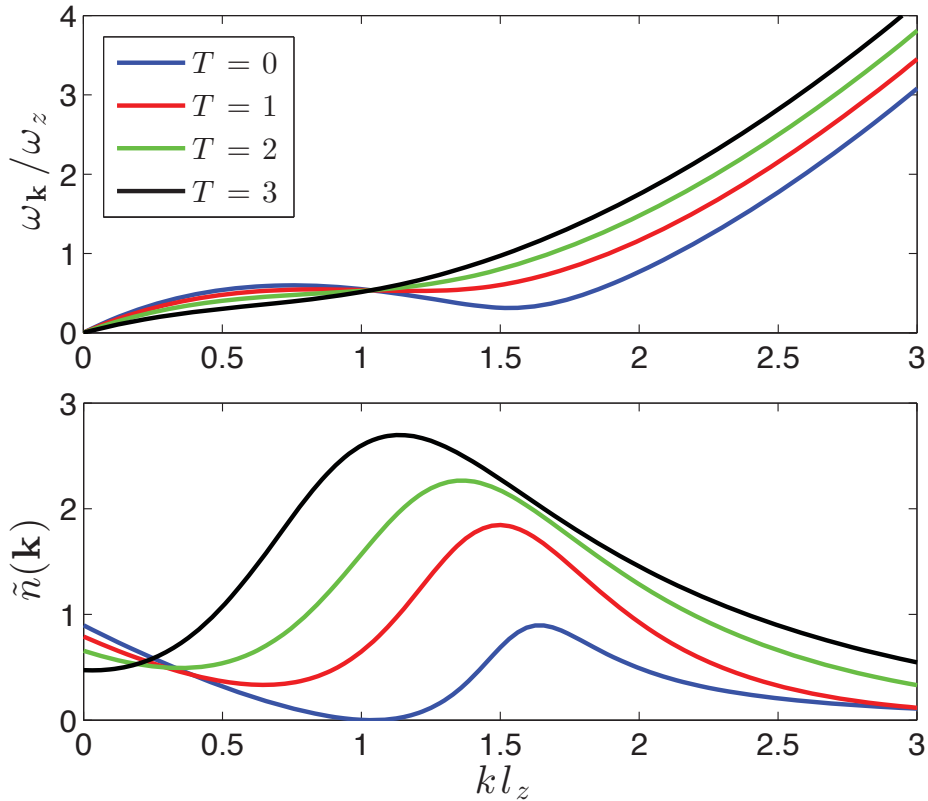
FIG. 8: Excitation spectrum of dipolar interactions at varying temperatures and occupation of the thermal cloud at varying temperatures and increasing momentum. As temperatures increase, the roton excitation requires more energy. With enough temperature, the roton excitation disappears entirely. The thermal cloud occupation reveals this feature by revealing a larger density of atoms at the roton momentum at zero temperature. As the temperature rises, the overall density of excitations increases as well as the majority of the excitations moves away from the former roton minimum towards lower momentum excitations.

IV. CONCLUSION

Particles at ultracold temperatures have two important features, their wavelike nature and whether the particles are bosons or fermions. Bosons, the particles we model, are integer spin particles that can occupy the same quantum space. Fermions, particles of half integer spin, cannot. We model boson-like atoms which consist of an even number of fermions, such as Sodium 23. As we cool a gas of bosons into the ultracold regime, we find the De Broglie

wavelength of the atoms begins to increase. Once the size of the wavelength becomes on the order of the inter particle spacing, a Bose Einstein condensate forms. This unique statistical phenomenon creates a new state of matter consisting of a macroscopic quantum mechanical state. Once the gas is cooled to absolute zero, every particle enters the BEC (only in the non-interacting case!).

Specifically, we modeled dipolar interactions within the BEC. However, dipolar interaction's attractive potential causes the condensate to collapse when in three dimensions so we confine our BEC to a nearly two dimensional geometry. As a result, the condensate gains moderate phase fluctuations, creating a quasi-condensate. We treat the quasi-condensate with nearly identical mathematical formalism, but still accounting for the phase fluctuations. The excitation spectrum for dipolar interactions in nearly two dimensions creates a peculiar excitation called a roton. The roton allows for high momentum excitations for very low energy. We investigate the extent to which rotons influence our condensate system.

Due to the Schrodinger's equation restrictions on symmetric wave functions and degrees of freedom, we find it prohibitively difficult to solve this problem. Therefore, we use an alternative method known as 2nd Quantization. We enumerate number states and apply creation and annihilation operators to them in order to count how many particles we have in each Fock state. We further make the mean field approximation, Hartree-Fock decomposition, Bogoliubov transformation, and finally the Popov approximation in order to create a diagonalized quadratic Hamiltonian.

We find that in nearly two dimensions, strong dipolar interactions cause a significant decrease in the superfluid fraction. In addition, two dimensions cause two phase transitions, first from a superfluid condensate to a non-superfluid condensate and second from a non-superfluid condensate to a thermal gas. When we calculate our solution self consistently, ensuring particle number stays constant, we find that the condensate will not collapse at zero temperature regardless of the dipolar interaction strength. While non-self consistent calculations cause the roton excitation energy to go to zero, exciting every atom out of the condensate, self consistent calculations reveal the roton energy never goes to zero. The system remains mechanically stable. In addition, raising the temperature of our system gradually removes the roton excitation, stabilizing the system. Our results will be revealing for modern experiments with ultra cold atoms focusing on dipolar interactions and the condensate.

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Appendix A: Derivation of Hartree-Fock Equations

We begin with the total Hamiltonian describing our system. We have our non-interacting component as the first term, which describes the collective kinetic energy of the system. Our second term consists of the potential term which will vary depending on which potential we introduce.

$$\hat{H} = \int \hat{\Psi}^\dagger(\vec{r})\hat{H}^1\hat{\Psi}(\vec{r})d\vec{r} + \frac{1}{2}\int \hat{\Psi}^\dagger(\vec{r})\hat{\Psi}^\dagger(\vec{r'})V(\vec{r}-\vec{r'})\hat{\Psi}(\vec{r})\hat{\Psi}(\vec{r'}) \tag{A1}$$

$\hat{\Psi}$ is a creation operator. It will create a particle at a certain position of $\vec{r}$. Using an orthonormal basis of plane waves, we set the creation operator $\hat{\Psi}$ to another operator $\hat{a}_{\vec{k}}$ which creates a particle of a certain momentum. An orthonormal basis is a basis of unit vectors which are all orthogonal to each other. An example would be $\hat{x}, \hat{y}$, and $\hat{z}$. They are three unit vectors which are orthogonal or perpendicular to each other.

$$\hat{\Psi}(\vec{r}) = \sum_{\vec{k}} \frac{1}{\sqrt{L^D}}e^{i\vec{k}\cdot\vec{r}}\hat{a}_{\vec{k}} \tag{A2}$$

where $\hat{a}_{\vec{k}}$ acts as an annihilation operator. $\hat{a}_{\vec{k}}^\dagger$ acts as a creation operator.

$$\hat{a}_{\vec{k}_i}^\dagger|N_{\vec{k}=0}, N_{\vec{k}_1}, ..., N_{\vec{k}_i}, ... > = \sqrt{N_{\vec{k}_i}+1}|N_{\vec{k}=0}, N_{\vec{k}_1}, ..., N_{\vec{k}_i}+1, ... >$$
$$\hat{a}_{\vec{k}_i}|N_{\vec{k}=0}, N_{\vec{k}_1}, ..., N_{\vec{k}_i}, ... > = \sqrt{N_{\vec{k}_i}}|N_{\vec{k}=0}, N_{\vec{k}_1}, ..., N_{\vec{k}_i}-1, ... > \tag{A3}$$

After substituting $\hat{\Psi}$ and $\hat{\Psi}^\dagger$ into the kinetic energy portion of the Hamiltonian, we find

$$\hat{H}_1 = \frac{1}{L^d}\sum_{\vec{k}\vec{q}}\int e^{-i\vec{k}\cdot\vec{r}}\left(-\frac{1}{2}\nabla^2\right)e^{i\vec{q}\cdot\vec{r}}\hat{a}_{\vec{k}}^\dagger\hat{a}_{\vec{q}}\ d\vec{r} \tag{A4}$$

The Laplace operator ∇^2 will return a value of $-q^2$. After moving this value outside the integral, our equation simplifies to

$$\hat{H}_1 = \frac{1}{2L^d}\sum_{\vec{k}\vec{q}}q^2\hat{a}_{\vec{k}}^\dagger\hat{a}_{\vec{q}}\int e^{-i\vec{k}\cdot\vec{r}}e^{i\vec{q}\cdot\vec{r}}d\vec{r} \tag{A5}$$

Because of our orthonormal basis, the $\vec{k}$ and $\vec{q}$ vectors are perpendicular and make the above integral zero, except when $\vec{k} = \vec{q}$. When both momentums are equal, the integral instead goes to L^d.

$$\hat{H}_1 = \frac{1}{2} \sum_{\vec{k}} k^2 \hat{a}_{\vec{k}}^\dagger \hat{a}_{\vec{k}} \tag{A6}$$

We follow the same process for the interaction potential, with $V(\vec{r} - \vec{r'}) = \sum_{\vec{p}} \frac{1}{L^d} \widetilde{V}(\vec{p}) e^{-i\vec{p}\cdot(\vec{r}-\vec{r'})}$. By following the same process above, we find

$$\hat{H}_{int} = \frac{1}{2L^d} \sum_{\vec{k}\vec{q}\vec{p}} \widetilde{V}(\vec{p}) \hat{a}_{\vec{p}+\vec{q}}^\dagger \hat{a}_{\vec{k}-\vec{q}}^\dagger \hat{a}_{\vec{k}} \hat{a}_{\vec{p}} \tag{A7}$$

At ultracold temperatures, bosonic gas forms a Bose Einstein Condensate, where the ground single particle state becomes macroscopically occupied by N_0 atoms. The result of ground state operators ($\vec{k} = 0$) on our Fock states ($|N_0...\rangle$) is approximately equal to $\sqrt{N_0}$ times our original Fock state. Thus, we can introduce the mean field approximation found below.

$$\langle a_0 \rangle \simeq \langle a_0^\dagger \rangle = \sqrt{N_0} \tag{A8}$$

We introduce this approximation to our Hamiltonian in A5 and A6. We expand the Hamiltonian in the form $\hat{H} = H_0 + \hat{H}_1 + \hat{H}_2 + ...$ where each term has 0, 1, or 2 non-condensate (non-ground state) operators. Noting that $\vec{k} = 0$, and introducing the condensate density of $n_0 = N_0/L^D$, we find

$$H_0 \equiv E_0 = \frac{N_0}{2} n_0 \widetilde{V}_0. \tag{A9}$$

This term is the condensate energy because it describes all energy within the condensate. The next term H_1 is neglected because terms with 1 operator will vanish when their expectation value is taken. Next we move to work on H_2. There are six possible ways to set two operators to $\vec{k} = 0$. As an example, the first two operators will be set to zero to

show the process for the mean field approximation. We set $q \to -p$ and $k \to p$. Our first two operators now equal $a_0 = \sqrt{N_0}$ and can be brought out front. Our resulting term is $\frac{N_0}{2L^D} \sum_{\vec{p}} \widetilde{V}(\vec{p}) \, \hat{a}^\dagger_{-\vec{p}} \hat{a}_{\vec{p}}$. After making a shift in quantum numbers $(p \to -k)$, our new term equals

$$\frac{N_0}{2L^D} \sum_{\vec{k}} \widetilde{V}(\vec{k}) \, \hat{a}^\dagger_{\vec{k}} \hat{a}_{-\vec{k}} \, . \tag{A10}$$

By setting each possible combination of operators to zero and forcing our equations to use similar momentum numbers, we find the result

$$\hat{H}_2 = \sum_{\vec{k} \neq 0} \left(\epsilon_{\vec{k}} - \mu \right) \hat{a}^\dagger_{\vec{k}} \hat{a}_{\vec{k}} + \frac{n_0}{2} \sum_{\vec{k} \neq 0} \left[2\widetilde{V}(0) \hat{a}^\dagger_{\vec{k}} \hat{a}_{\vec{k}} + \widetilde{V}(\vec{k}) \left(2\hat{a}^\dagger_{\vec{k}} \hat{a}_{\vec{k}} + \hat{a}_{\vec{k}} \hat{a}_{-\vec{k}} + \hat{a}^\dagger_{\vec{k}} \hat{a}^\dagger_{-\vec{k}} \right) \right] . \tag{A11}$$

The Hamiltonian above does not conserve particle number so we must introduce a Lagrange multiplier, the chemical potential, to keep particle number constrained. We place the chemical potential, μ as a constraint on ϵ.

The next term in our Hamiltonian expansion is H_3, but just at H_1 went to zero, H_3 also will. Odd numbers of raising and lowering operators, regardless of order or capacity, will vanish when an expectation value is calculated. The final component of the Hamiltonian, H_4, is quartic. At this point, the rest of the Hamiltonian is quadratic so we need to force H_4 to be quadratic as well. We use Hartree-Fock decomposition on H_4 to accomplish this. We take each combination of 2 terms and take an expectation value of them.

$$\hat{H}_4 = \frac{1}{2L^d} \sum_{\vec{k}, \vec{q}, \vec{p} \neq 0} \widetilde{V}(\vec{p}) \hat{a}^\dagger_{\vec{p}+\vec{q}} \hat{a}^\dagger_{\vec{k}-\vec{q}} \hat{a}_{\vec{k}} \hat{a}_{\vec{p}} \tag{A12}$$

$$\hat{H}_4 = \sum_{\vec{k}, \vec{q}, \vec{p} \neq 0} \frac{\widetilde{V}(\vec{p})}{2L^d} \left[\hat{a}^\dagger_{\vec{p}+\vec{q}} \hat{a}^\dagger_{\vec{k}-\vec{q}} \langle \hat{a}_{\vec{k}} \hat{a}_{\vec{p}} \rangle + \hat{a}^\dagger_{\vec{p}+\vec{q}} \hat{a}_{\vec{p}} \langle \hat{a}^\dagger_{\vec{k}-\vec{q}} \hat{a}_{\vec{k}} \rangle + \hat{a}^\dagger_{\vec{p}+\vec{q}} \hat{a}_{\vec{k}} \langle \hat{a}^\dagger_{\vec{k}-\vec{q}} \hat{a}_{\vec{p}} \rangle \right.$$

$$\left. + \hat{a}^\dagger_{\vec{k}-\vec{q}} \hat{a}_{\vec{p}} \langle \hat{a}^\dagger_{\vec{p}+\vec{q}} \hat{a}_{\vec{k}} \rangle + \hat{a}^\dagger_{\vec{k}-\vec{q}} \hat{a}_{\vec{k}} \langle \hat{a}^\dagger_{\vec{p}+\vec{q}} \hat{a}_{\vec{p}} \rangle + \hat{a}_{\vec{k}} \hat{a}_{\vec{p}} \langle \hat{a}^\dagger_{\vec{p}+\vec{q}} \hat{a}^\dagger_{\vec{k}-\vec{q}} \rangle \right] \tag{A13}$$

We introduce two densities into this equation now. The first is called the normal density $\tilde{n}_k$ and the second is called the anomalous density $\tilde{m}_k$.

$$\tilde{n}_k = \langle \hat{a}_{\vec{k}}^\dagger \hat{a}_{\vec{k}} \rangle \tag{A14}$$

$$\tilde{m}_k = \langle \hat{a}_{\vec{k}} \hat{a}_{-\vec{k}} \rangle = \langle \hat{a}_{\vec{k}}^\dagger \hat{a}_{-\vec{k}}^\dagger \rangle \tag{A15}$$

In order to substitute the densities into $\hat{H}_4$, we create equations to swap our quantum numbers. For our first term within the bracket, we set $\vec{p} \to -\vec{k}$ resulting in $\hat{a}_{\vec{q}-\vec{k}}^\dagger \hat{a}_{\vec{k}-\vec{q}}^\dagger \frac{\widetilde{V}(\vec{q})}{2L^D} \tilde{m}_k$. Following, we set $\vec{p} \to \vec{k} - \vec{q}$. This action results in $\hat{a}_{\vec{p}}^\dagger \hat{a}_{-\vec{p}}^\dagger \frac{\widetilde{V}(\vec{k}-\vec{p})}{2L^D} \tilde{m}_k$. In order to get this term into a common form with the other five terms, we swap the quantum momentum numbers $\vec{p} \leftrightarrow \vec{k}$. Finally, our first term is in the desired form below.

$$Term\, 1 = \hat{a}_{\vec{k}}^\dagger \hat{a}_{-\vec{k}}^\dagger \frac{\widetilde{V}(\vec{p} - \vec{k})}{2L^D} \tilde{m}_p \tag{A16}$$

Term 2 will use a normal density as oppose to the anomalous density used in term 1. In order to alter term 2 into the desired form, we set $\vec{q} = 0$ resulting in $\hat{a}_{\vec{p}}^\dagger \hat{a}_{\vec{p}} \langle \hat{a}_{\vec{k}}^\dagger \hat{a}_{\vec{k}} \rangle \widetilde{V}(0)$. After substituting the normal density and allowing for a quantum number swap $\vec{p} \leftrightarrow \vec{k}$.

We follow this same procedure for the remaining four terms. All process similar to those described above. Our new $\hat{H}_4$ follows below

$$\hat{H}_4 = \tilde{n} \widetilde{V}(0) \sum_{\vec{k} \neq 0} \hat{a}_{\vec{k}}^\dagger \hat{a}_{\vec{k}} + \frac{1}{2L^D} \sum_{\vec{k}\vec{q} \neq 0} \widetilde{V}(\vec{q} - \vec{k})(\tilde{n}(q)\hat{a}_{\vec{k}}^\dagger \hat{a}_{\vec{k}} + \tilde{m}(q)\hat{a}_{\vec{k}}^\dagger \hat{a}_{\vec{k}}). \tag{A17}$$

$$\hat{H}_2 = \sum_{\vec{k} \neq 0} (\epsilon_{\vec{k}} - \mu)\hat{a}_{\vec{k}}^\dagger \hat{a}_{\vec{k}} + \frac{n_0}{2} \sum_{\vec{k} \neq 0} \left[2\widetilde{V}(0)\hat{a}_{\vec{k}}^\dagger \hat{a}_{\vec{k}} + \widetilde{V}(\vec{k}) \left(2\hat{a}_{\vec{k}}^\dagger \hat{a}_{\vec{k}} + \hat{a}_{\vec{k}} \hat{a}_{-\vec{k}} + \hat{a}_{\vec{k}}^\dagger \hat{a}_{-\vec{k}}^\dagger \right) \right]. \tag{A18}$$

We take $\hat{H}_2$ and add it to $\hat{H}_4$ to form the Hamiltonian for non-condensate atoms interacting with non-condensate atoms.

$$\hat{H}_2' = \sum_{\vec{k} \neq 0} (\epsilon_{\vec{k}} - \mu + \Sigma_{11}(\vec{k}))\hat{a}_{\vec{k}}^\dagger \hat{a}_{\vec{k}} + \frac{1}{2} \sum_{\vec{k} \neq 0} \left(\Sigma_{20}(\vec{k})\hat{a}_{\vec{k}} \hat{a}_{-\vec{k}} + \Sigma_{02}^*(\vec{k})\hat{a}_{\vec{k}}^\dagger \hat{a}_{-\vec{k}}^\dagger \right). \tag{A19}$$

We find the above equation by selectively defining Σ_{11} and Σ_{20}.

$$\Sigma_{11}(\vec{k}) = \tilde{n}\widetilde{V}(0) + \frac{1}{L^D}\tilde{n}(\vec{q})\widetilde{V}(\vec{q}-\vec{k}) + n_0\widetilde{(0)} + n_0\widetilde{V}(\vec{k}) \tag{A20}$$

After combining terms, we find

$$\Sigma_{11}(\vec{k}) = n\widetilde{V}(0) + n_0\widetilde{V}(\vec{k}) + \frac{1}{L^D}\sum_{\vec{q}\neq 0}\tilde{n}(\vec{q})\widetilde{V}(\vec{q}-\vec{k}) \tag{A21}$$

.

Similarly, we selectively choose $\Sigma_{20} = \Sigma_{02}^*$ to equal the terms associated with $\hat{a}_{\vec{k}}\hat{a}_{-\vec{k}}$ and $\hat{a}_{\vec{k}}^\dagger\hat{a}_{-\vec{k}}^\dagger$. After minimal simplification,

$$\Sigma_{20}(\vec{k}) = n_0\widetilde{V}(\vec{k}) + \frac{1}{L^D}\sum_{\vec{q}\neq 0}\tilde{m}(\vec{q})\widetilde{V}(\vec{q}-\vec{k}) \tag{A22}$$

.

Appendix B: Expansion via the Bogoliubov Transformation

In order to diagonalize our Hamiltonian, we use the Bogoliubov Transformation. Bogoliubov's process redefines our creation and annihilation as follows

$$\hat{a}_{\vec{k}} = u_{\vec{k}}\hat{b}_{\vec{k}} + v_{\vec{k}}^*\hat{b}_{-\vec{k}}^\dagger \tag{B1}$$

$$\hat{a}_{\vec{k}}^\dagger = u_{\vec{k}}^*\hat{b}_{\vec{k}}^\dagger + v_{\vec{k}}\hat{b}_{-\vec{k}} \tag{B2}$$

Using these relations, we also redefine our commutation relation in new raising and lowering operators.

$$\left[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}^\dagger\right] = \delta_{kk'} \tag{B3}$$

$$\left[u_{\vec{k}}\hat{b}_{\vec{k}} + v_{\vec{k}}^*\hat{b}_{-\vec{k}}^\dagger, u_{\vec{k}'}^*\hat{b}_{\vec{k}'}^\dagger + v_{\vec{k}'}\hat{b}_{-\vec{k}'}\right] = \delta_{kk'} \tag{B4}$$

$$\left[u_{\vec{k}}\hat{b}_{\vec{k}}, u_{\vec{k}'}^*\hat{b}_{\vec{k}'}^\dagger\right] + \left[u_{\vec{k}}\hat{b}_{\vec{k}}, v_{\vec{k}'}\hat{b}_{-\vec{k}'}\right] + \left[v_{\vec{k}}^*\hat{b}_{-\vec{k}}^\dagger, u_{\vec{k}'}^*\hat{b}_{\vec{k}'}^\dagger\right] + \left[v_{\vec{k}'}^*\hat{b}_{-\vec{k}'}^\dagger, v_{\vec{k}'}\hat{b}_{-\vec{k}'}\right] = \delta_{kk'} \tag{B5}$$

By definition,

$$\left[u_{\vec{k}}\hat{b}_{\vec{k}}, v_{\vec{k}'}\hat{b}_{-\vec{k}'}\right] = \left[v_{\vec{k}}^*\hat{b}_{-\vec{k}}^\dagger, u_{\vec{k}'}^*\hat{b}_{\vec{k}'}^\dagger\right] = 0. \tag{B6}$$

$$u_{\vec{k}}u_{\vec{k}'}^*\left[\hat{b}_{\vec{k}}, \hat{b}_{\vec{k}'}^\dagger\right] + v_{\vec{k}}^*v_{\vec{k}'}\left[\hat{b}_{-\vec{k}}^\dagger, \hat{b}_{-\vec{k}'}\right] = \delta_{kk'} \tag{B7}$$

$$u_{\vec{k}}u_{\vec{k}'}^*\delta_{kk'} - v_{\vec{k}}^*v_{\vec{k}'}\delta_{kk'} = \delta_{kk'} \tag{B8}$$

We set $\vec{k} = \vec{k}'$, forcing the Dirac delta functions equal 1. This results in a normalization constraint for $u_{\vec{k}}$ and $v_{\vec{k}}$.

$$u_{\vec{k}}u_{\vec{k}}^* - v_{\vec{k}}^*v_{\vec{k}} = 1 \tag{B9}$$

Now we substitute our redefined creation and annihilation operators into our Hamiltonian. For ease of explanation, we will only show $\hat{H}_2$ with hastened simplifications.

$$\hat{H}_{0,2} = \frac{(N_0)^2 \widetilde{V}(0)}{2L^D} + \frac{N_0}{2} \sum_{\vec{k}} \left[\widetilde{V}(\vec{k}) \left(\hat{a}^\dagger_{\vec{k}} \hat{a}^\dagger_{-\vec{k}} + \hat{a}_{\vec{k}} \hat{a}_{-\vec{k}} + 2\hat{a}^\dagger_{\vec{k}} \hat{a}_{\vec{k}} \right) + 2\widetilde{V}(0)\hat{a}^\dagger_{\vec{k}} \hat{a}_{\vec{k}} \right] \qquad (B10)$$

$$\hat{H}_{0,2} = \frac{(N_0)^2 \widetilde{V}(0)}{2L^D} + \frac{N_0}{2} \sum_{\vec{k}} \left[\widetilde{V}(\vec{k}) \left((u^*_{\vec{k}} \hat{b}^\dagger_{\vec{k}} + v_{\vec{k}} \hat{b}_{-\vec{k}})(u^*_{\vec{k}} \hat{b}^\dagger_{\vec{k}} + v_{\vec{k}} \hat{b}_{-\vec{k}}) + (u_{\vec{k}} \hat{b}_{\vec{k}} + v^*_{\vec{k}} \hat{b}^\dagger_{-\vec{k}})(u_{\vec{k}} \hat{b}_{\vec{k}} + v^*_{\vec{k}} \hat{b}^\dagger_{-\vec{k}}) \right. \right.$$

$$\left. \left. + 2(u^*_{\vec{k}} \hat{b}^\dagger_{\vec{k}} + v_{\vec{k}} \hat{b}_{-\vec{k}})(u_{\vec{k}} \hat{b}_{\vec{k}} + v^*_{\vec{k}} \hat{b}^\dagger_{-\vec{k}}) \right) + 2\widetilde{V}(0)(u^*_{\vec{k}} \hat{b}^\dagger_{\vec{k}} + v_{\vec{k}} \hat{b}_{-\vec{k}})(u_{\vec{k}} \hat{b}_{\vec{k}} + v^*_{\vec{k}} \hat{b}^\dagger_{-\vec{k}}) \right] \qquad (B11)$$

We multiply out all raising and lowering operators, combining like terms. We remove the off diagonal terms and set them equal to zero. We also assume all our values for $\vec{k}$ are real causing $-\vec{k} = \vec{k}$

$$\left[2(u_{\vec{k}} u_{\vec{k}} + v_{\vec{k}} v_{\vec{k}} + 2u_{\vec{k}} v_{\vec{k}})\widetilde{V}(\vec{k}) + (2u_{\vec{k}} v_{\vec{k}})\widetilde{V}(0) + u_{\vec{k}} v_{\vec{k}}(\epsilon_{\vec{k}} - \mu) \right] \vec{b}_{\vec{k}} \vec{b}_{\vec{k}} = 0 \qquad (B12)$$

The term for $\hat{b}^*_{\vec{k}} \hat{b}^*_{\vec{k}}$ is simply the complex conjugate of the above term. However, since both terms are real, the resulting term will only be a conjugate.

We need to add the single particle energy term, $\epsilon_{\vec{k}}$, into our Hamiltonian to ensure we have the full spectrum of energy (interaction potential and kinetic energy). By combining the commutation relation $u^2_{\vec{k}} - v^2_{\vec{k}} = 1$ and the off diagonal terms, we can solve for the values of $u_{\vec{k}}$ and $v_{\vec{k}}$. We utilize Mathematica to solve and partially simplify. Through additional algebra steps, we find

$$u_{\vec{k}} = \frac{1}{\sqrt{2}} \sqrt{1 + \frac{\widetilde{V}(\vec{k}) + \epsilon_{\vec{k}} - \mu}{\omega_{\vec{k}}}} \qquad (B13)$$

where

$$\omega_{\vec{k}} = \epsilon_{\vec{k}}^{\frac{1}{2}} (2\widetilde{V}(\vec{k}) + \epsilon_{\vec{k}})^{\frac{1}{2}} \qquad (B14)$$

.

$$v_{\vec{k}} = -sgn\widetilde{V}(\vec{k})\frac{1}{\sqrt{2}}\sqrt{\frac{\widetilde{V}(\vec{k}) + \epsilon_{\vec{k}} - \mu}{\omega_{\vec{k}}} - 1} \qquad (B15)$$

The remaining diagonal terms are show in their entirety below

$$\hat{H}_{0,2} = \frac{(N_0)^2\widetilde{V}(0)}{2L^D} + \frac{N_0}{2}\sum_{\vec{k}}\left[\widetilde{V}(\vec{k})\left(u_{\vec{k}}^*v_{\vec{k}}\hat{b}_{\vec{k}}^\dagger\hat{b}_{\vec{k}} + v_{\vec{k}}u_{\vec{k}}^*\hat{b}_{\vec{k}}\hat{b}_{\vec{k}}^\dagger + u_{\vec{k}}v_{\vec{k}}^*\hat{b}_{\vec{k}}\hat{b}_{\vec{k}}^\dagger + v_{\vec{k}}^*u_{\vec{k}}\hat{b}_{\vec{k}}^\dagger\hat{b}_{\vec{k}} + 2u_{\vec{k}}^*u_{\vec{k}}\hat{b}_{\vec{k}}^\dagger\hat{b}_{\vec{k}} + \right.\right.$$
$$\left.\left. 2v_{\vec{k}}v_{\vec{k}}^*\hat{b}_{\vec{k}}\hat{b}_{\vec{k}}^\dagger\right) + 2\widetilde{V}(0)\left(u_{\vec{k}}^*u_{\vec{k}}\hat{b}_{\vec{k}}^\dagger\hat{b}_{\vec{k}} + v_{\vec{k}}v_{\vec{k}}^*\hat{b}_{\vec{k}}\hat{b}_{\vec{k}}^\dagger\right) + (u_{\vec{k}}^*u_{\vec{k}} + v_{\vec{k}}v_{\vec{k}*}(\epsilon_{\vec{k}} - \mu))\right]. \qquad (B16)$$

We replace $\hat{b}_{\vec{k}}\hat{b}_{\vec{k}}^\dagger$ with $\hat{b}_{\vec{k}}^\dagger\hat{b}_{\vec{k}} + 1$ as allowed by the commutation relation. In addition, we remove all * because we assume $u_{\vec{k}}$ and $v_{\vec{k}}$ are real. After algebraic simplification we find

$$\hat{H}_{0,2} = \frac{(N_0)^2\widetilde{V}(0)}{2L^D} + \frac{N_0}{2}\sum_{\vec{k}}\left[\widetilde{V}(\vec{k})\left(u_{\vec{k}}^2 + 2u_{\vec{k}}v_{\vec{k}} + v_{\vec{k}}^2\right)\hat{b}_{\vec{k}}^\dagger\hat{b}_{\vec{k}} + 2\widetilde{V}(0)(u_{\vec{k}}^2 + v_{\vec{k}}^2)\hat{b}_{\vec{k}}^\dagger\hat{b}_{\vec{k}} + \right.$$
$$\left.\widetilde{V}(\vec{k})(2u_{\vec{k}}v_{\vec{k}} + v_{\vec{k}}^2) + 2\widetilde{V}(0)v_{\vec{k}}^2(u_{\vec{k}}^2 + v_{\vec{k}}^2)(\epsilon_{\vec{k}} - \mu)\hat{b}_{\vec{k}}^\dagger\hat{b}_{\vec{k}} + u_{\vec{k}}v_{\vec{k}}(\epsilon_{\vec{k}} - \mu)\right] \qquad (B17)$$

.

We proceed to plug in our values for $u_{\vec{k}}$ and $v_{\vec{k}}$. After simplifying using Mathematica, our Hamiltonian for $\hat{H}_{0,2}$ simplifies to

$$\hat{H}_{0,2} = \frac{(N_0)^2\widetilde{V}(0)}{2L^D} + N_0\sum_{\vec{k}}\omega_{\vec{k}}\hat{b}_{\vec{k}}^\dagger\hat{b}_{\vec{k}} + \frac{N_0}{2}\sum_{\vec{k}}\left(\omega_{\vec{k}} - \left(\epsilon_{\vec{k}} - \mu\right) - n_0\widetilde{V}(\vec{k})\right) \qquad (B18)$$

.

Appendix C: Number Operator in terms of Creation and Annihilation Operators

In our Hamiltonian, we find pairs of quadratic creation and annihilation operators in the form $\hat{a}_{\vec{k}}^{\dagger} \hat{a}_{\vec{k}}$. This acts as a number operator $\hat{N}$, an operator which counts the number of particles in a given state.

In order to prove $\hat{a}_{\vec{k}}^{\dagger} \hat{a}_{\vec{k}}$ acts as a number operator, we cat on the following system $|E_0, E_1, E_2, E_3, E_4, ... >$.

$$\hat{N}_{k=2}|k_0, k_1, k_2, k_3, k_4, ... > = \hat{a}_{\vec{k}_2}^{\dagger} \hat{a}_{\vec{k}_2}|k_0, k_1, k_2, k_3, k_4, ... > \tag{C1}$$

$$\hat{a}_{\vec{k}_2}^{\dagger} \hat{a}_{\vec{k}_2}|k_0, k_1, k_2, k_3, k_4, ... > = \sqrt{k_2}\hat{a}_{\vec{k}_2}^{\dagger}|k_0, k_1, k_2 - 1, k_3, k_4, ... > \tag{C2}$$

$$\sqrt{k_2}\hat{a}_{\vec{k}_2}^{\dagger}|k_0, k_1, k_2, k_3, k_4, ... > = \sqrt{k_2}\sqrt{k_2 - 1 + 1}|k_0, k_1, k_2, k_3, k_4, ... > \tag{C3}$$

$$\hat{N}_{k=2} = k_2|k_0, k_1, k_2, k_3, k_4, ... > \tag{C4}$$

We can apply this knowledge to examining a non-interacting group of particles. We have a system of four particles

$$\chi_{\vec{k}} = |k_0, k_1, k_2, k_3, k_4 > = |0, 1, 0, 2, 1 > \tag{C5}$$

When we determine the energy for this system, we apply the Hamiltonian for a non-interacting system.

$$\hat{H} = \sum_{\vec{k}} \hat{a}_{\vec{k}}^{\dagger} \hat{a}_{\vec{k}} \epsilon_{\vec{k}} \tag{C6}$$

After counting each individual states kinetic energy and summing over all the states, we find our collective energy.

$$\hat{H}\chi_{\vec{k}} = E\chi_{\vec{k}} = \left(\frac{k_1^2}{2} + \frac{2k_3^2}{2} + \frac{k_4^2}{2} \right)|0, 1, 0, 2, 1 > \tag{C7}$$

Appendix D: Derivation of Quasi-2D Contact Interaction Potential

We begin with the general equation for an interacting Hamiltonian.

$$\hat{H} = \int \hat{\Psi}^\dagger(\vec{r})\hat{H}^1\hat{\Psi}(\vec{r})d\vec{r} + \frac{1}{2}\int \hat{\Psi}^\dagger(\vec{r})\hat{\Psi}^\dagger(\vec{r'})V(\vec{r}-\vec{r'})\hat{\Psi}(\vec{r})\hat{\Psi}(\vec{r'}) \tag{D1}$$

Then we make the following substitutions into our general interacting Hamiltonian.

$$\hat{\Psi}(\vec{r}) = \sum_{\vec{k}} \chi(z)\frac{1}{\sqrt{L^D}}e^{i\vec{k}\cdot\vec{\rho}}\hat{a}_{\vec{k}} \tag{D2}$$

$$\hat{\Psi}^\dagger(\vec{r}) = \sum_{\vec{k}} \chi^*(z)\frac{1}{\sqrt{L^D}}e^{-i\vec{k}\cdot\vec{\rho}}\hat{a}_{\vec{k}}^\dagger \tag{D3}$$

$$\chi(z) = \chi^*(z) = \frac{1}{\sqrt{l_z}\pi^{\frac{1}{4}}}e^{\frac{-z^2}{2l_z^2}} \tag{D4}$$

$$V(\vec{r}-\vec{r'}) = \sum_{\vec{p}} \widetilde{V}(\rho)\frac{1}{L^D}e^{-i\vec{p}\cdot(\vec{r}-\vec{r'})} \tag{D5}$$

We begin with calculating the simpler non-interacting component and then follow with the interacting portion.

$$H_1 = \int d\vec{r}\chi^*(z)\frac{1}{L}e^{-i\vec{k}\cdot\vec{\rho}}(-\frac{1}{2}\nabla^2 - U(z))\frac{1}{L}e^{i\vec{q}\cdot\vec{\rho}}\chi(z)\hat{a}_{\vec{k}}^\dagger\hat{a}_{\vec{q}} \tag{D6}$$

$$H_1 = \sum_{\vec{k}\vec{q}} \frac{1}{L^2}\int d\vec{r}\chi(z)e^{-i\vec{k}\cdot\vec{\rho}}(k^2 - \frac{1}{2}\frac{d^2}{dz^2} + \frac{1}{2}z^2)\frac{1}{L}e^{i\vec{q}\cdot\vec{\rho}}\chi(z)\hat{a}_{\vec{k}}^\dagger\hat{a}_{\vec{q}} \tag{D7}$$

The Hamiltonian operator within the non-interacting portion provides an average kinetic energy summed over all particles. This gives us the total kinetic energy of the system. The potential portion is the same as for a harmonic oscillator and since $\chi(z)$ is the ground state of the system, the potential is simply $\frac{1}{2}\hbar\omega$. Since we set the condition that $\hbar = m = \omega = 1$ as an assumption to simplify calculations, this leaves a potential of $\frac{1}{2}$

$$H_1 = \frac{1}{L^2} \sum_{\vec{k}\vec{q}} \left(\frac{k^2}{2} + \frac{1}{2} \right) \int \chi(z) e^{-i(\vec{k}-\vec{q})\cdot\vec{\rho}} d\vec{\rho} \int \chi^2(z) d\vec{z} \; \hat{a}_{\vec{k}}^\dagger \hat{a}_{\vec{q}} \tag{D8}$$

$$H_1 = \sum_{\vec{k}} \left(\frac{k^2}{2} + \frac{1}{2} \right) \hat{a}_{\vec{k}}^\dagger \hat{a}_{\vec{q}} \tag{D9}$$

Since $\frac{1}{2}$ is a constant, it can be ignored since it affects every state equally. Therefore, our final non interacting component is the exact same as our 3-D case.

$$H_1 = \sum_{\vec{k}} \frac{k^2}{2} \hat{a}_{\vec{k}}^\dagger \hat{a}_{\vec{q}} \tag{D10}$$

Now we continue on to the more interesting 2-D contact interaction term.

$$H_{int} = \frac{1}{2} \int \hat{\Psi}^\dagger(\vec{r}) \hat{\Psi}^\dagger(\vec{r'}) V(\vec{r} - \vec{r'}) \hat{\Psi}(\vec{r}) \hat{\Psi}(\vec{r'}) \tag{D11}$$

$$= \frac{1}{2} \int \int d\vec{r} d\vec{r'} \chi^2(z) \chi^2(z') \frac{1}{L^4} \sum_{\vec{p}} \widetilde{V}(p) \frac{1}{L^2} e^{-i\vec{p}(\vec{r}-\vec{r'})} e^{i\vec{k}\cdot\vec{p}} e^{i\vec{q}\cdot\vec{p}} e^{-i\vec{k'}\cdot\vec{p}} e^{-i\vec{q'}\cdot\vec{p}} \hat{a}_{\vec{k}} \hat{a}_{\vec{q}} \hat{a}_{\vec{k'}}^\dagger \hat{a}_{\vec{q'}}^\dagger \tag{D12}$$

$$= \frac{1}{2} \int \int d\vec{r} d\vec{r'} \chi^2(z) \chi^2(z') \frac{1}{L^6} \sum_{\vec{p_{2D}}} \widetilde{V}(p) e^{-i\vec{p}(\vec{\rho}-\vec{\rho'})} e^{i\vec{p}(\vec{k}+\vec{q}-\vec{k'}-\vec{q'})} \sum_{\vec{p_z}} e^{-i\vec{p_z}(z-z')} \hat{a}_{\vec{k}} \hat{a}_{\vec{q}} \hat{a}_{\vec{k'}}^\dagger \hat{a}_{\vec{q'}}^\dagger \tag{D13}$$

$$= \frac{1}{2} \int \int dz dz' n_z(z) n_z(z) \frac{1}{L^2} \delta(k+q-p) \delta(p'-k'-q') \int \frac{dk_z}{2\pi} \widetilde{V}(p) e^{-i\vec{p_z}(z-z')} \hat{a}_{\vec{k}} \hat{a}_{\vec{q}} \hat{a}_{\vec{k'}}^\dagger \hat{a}_{\vec{q'}}^\dagger \tag{D14}$$

$$H_{int} = \frac{1}{2L} \int dz e^{-i\vec{p_z}} n_z(z) \int dz' e^{i\vec{p_z'}} n_z(z') \int \frac{dk_z}{2\pi} \widetilde{V}(p) e^{-i\vec{p_z}(z-z')} \hat{a}_{\vec{p}+\vec{q}} \hat{a}_{\vec{k}-\vec{q}} \hat{a}_{\vec{p}}^\dagger \hat{a}_{\vec{k}}^\dagger \tag{D15}$$

Here we use the Fourier transform (or momentum space equivalent) $\tilde{n}_z(k_z) = \int dz e^{ik_z \cdot z} \chi^2(z)$ in order to change the equation into 2D momentum space.

$$H_{int} = \frac{1}{2L} \int \frac{dk_z}{2\pi} \widetilde{V}(p) \tilde{n}_z(k_z) \tilde{n}_z(k_z) \hat{a}_{\vec{p}+\vec{q}} \hat{a}_{\vec{k}-\vec{q}} \hat{a}_{\vec{p}}^\dagger \hat{a}_{\vec{k}}^\dagger \tag{D16}$$

$$H_{int} = \frac{1}{2} \sum_{\vec{k}\vec{q}\vec{p}} \hat{a}_{\vec{p}+\vec{q}} \hat{a}_{\vec{k}-\vec{q}} \hat{a}_{\vec{p}}^\dagger \hat{a}_{\vec{k}}^\dagger \widetilde{V}_{q2D}(\vec{p}) \tag{D17}$$

Appendix E: Derivation of Quasi-2D Dipolar Interaction Potential

The difference between the contact and dipolar interaction potential lies only in the the potential term. Dipolar interactions follow the interaction potential $V_d(\vec{x}) = d^2 \frac{1-3\cos^2\theta}{x^3}$. To approach this problem, we will use spherical harmonics and Bessel functions.

$$H_{int} = \frac{1}{2} \int \hat{\Psi}^\dagger(\vec{r})\hat{\Psi}^\dagger(\vec{r'})V(\vec{r}-\vec{r'})\hat{\Psi}(\vec{r})\hat{\Psi}(\vec{r'})d\vec{r}d\vec{r'} \tag{E1}$$

Below is the potential for dipolar interactions

$$V_d(\vec{x}) = d^2 \frac{1-3\cos^2(\theta)}{x^3} \tag{E2}$$

We expand the potential using spherical harmonics.

$$V_d(\vec{r}) = \frac{d^2}{(\vec{r})^3}4\sqrt{\frac{\pi}{5}}Y_{20} \tag{E3}$$

We then Fourier transform and spherically expand our result into a 3D momentum space potential. What remains to be done is to convert our 3D potential into a quasi 2D potential.

$$e^{i\vec{k}\cdot\vec{r}} = 4\pi \sum_{l,m} i^l Y_{lm}^*(\hat{k})j_l(kx)Y_{lm}(\hat{x}) \tag{E4}$$

$$\widetilde{V}_d(\vec{k}) = \pi d^2 \frac{4}{3}(3\cos^2(\theta_k)-1) \qquad where \ \ \cos(\theta_k) = \frac{k_z}{|k|} \tag{E5}$$

$$\widetilde{V}_{q2Dd}(\vec{k}) = \int \frac{dk_z}{2\pi}\tilde{n}_z^2(k_z)\widetilde{V}_d(\vec{k}) \tag{E6}$$

After substituting our value for $\widetilde{V}_k(\vec{k})$ into the equation we calculated above in Appendix A, we use Mathematica to calculate $\widetilde{V}_{q2Dd}(\vec{k})$.

$$\widetilde{V}_{q2Dd}(\vec{k}) = \frac{g_d}{\sqrt{2\pi}l_z}\left(2 - 3\sqrt{\pi}qe^{q^2}erfc[q]\right) \qquad where \ \ q = \frac{\vec{k}_\rho l_z}{\sqrt{2}} \tag{E7}$$

g_d is a substitution used to give the equation a similar form regardless of our interaction type. By making $g_d = \frac{4\pi\hbar^2 a_{dd}}{m}$, $\widetilde{V}_k(\vec{k})$ inhabits the same form whether in the 2D contact case or the 2D dipolar case. In this equation, $a_{dd} = \frac{md^2}{3\hbar^2}$. $\widetilde{V}_{q2Dd}(\vec{k})$ has a crucial distinction between it and the 2D contact case. The momentum dependence coinciding with dipolar interactions makes calculations significantly more difficult than with momentum independent potentials.

Appendix F: Solving the Hartree Fock Bogoliubov Equations

Through our mathematical framework, we find a collection of equations we call the Hartree Fock Bogoliubov (HFB) equations.

$$\mu = n\widetilde{V}(0) + \frac{1}{L^D} \sum_{\vec{k}} \widetilde{V}(\vec{k})\tilde{n}(\vec{k}) \tag{F1}$$

$$\Sigma_{11}(\vec{k}) = n\widetilde{V}(0) + n_0\widetilde{V}(\vec{k}) + \frac{1}{L^D} \sum_{\vec{q}\neq 0} \widetilde{n}(\vec{q})\widetilde{V}(\vec{q}-\vec{k}) \tag{F2}$$

$$\Sigma_{20}(\vec{k}) = n_0\widetilde{V}(\vec{k}) \tag{F3}$$

$$\omega_{\vec{k}} = \sqrt{\left(\epsilon_{\vec{k}} - \mu + \Sigma_{11}(\vec{k})\right)^2 - (\Sigma_{20}(\vec{k}))^2} \tag{F4}$$

$$u_{\vec{k}} = \sqrt{\frac{1}{2}\left(\frac{\epsilon_{\vec{k}} - \mu + \Sigma_{11}(\vec{k})}{\omega_{\vec{k}}} + 1\right)} \tag{F5}$$

$$v_{\vec{k}} = -\sqrt{\frac{1}{2}\left(\frac{\epsilon_{\vec{k}} - \mu + \Sigma_{11}(\vec{k})}{\omega_{\vec{k}}} - 1\right)} \tag{F6}$$

$$\widetilde{n}(\vec{k}) = \left(u_{\vec{k}}^2 + v_{\vec{k}}^2\right) f_{\vec{k}} + v_{\vec{k}}^2 \tag{F7}$$

$$f_{\vec{k}} = \frac{1}{e^{\beta\omega_{\vec{k}}} - 1} \tag{F8}$$

$$n = n_0 + \int \frac{d\vec{k}}{(2\pi)^3}\widetilde{n}(\vec{k}) \tag{F9}$$

We have nine equations and nine unknowns! In order to calculate a self consistent solution, we must ensure that particle number remains constant. We first guess a solution for the condensate fraction n_0. We use this value to calculate the self energies Σ_{11} and Σ_{20}. We use the self energies to calculate $\omega_{\vec{k}}$, $u_{\vec{k}}$, and $v_{\vec{k}}$. Once we have the value for $\omega_{\vec{k}}$, we plug it into the Bose Einstein distribution to find $f_{\vec{k}}$ which we use to find our thermal density, $\widetilde{n}(\vec{k})$. We use the thermal density to find the number of excited atoms in the system.

However, we often find the number of atoms in our system, excited atoms plus condensate atoms, exceed the total particle number. In order to fix this, we correct our initial guess of the condensate fraction and iterate the process using the Matlab code shown in Appendix G. Through several iterations, we find a converged solution. We perform this analysis to calculate all of our results.

Appendix G: Matlab Code for Solving Hartree Fock Bogoliubov Equations

Here we include the Matlab code used for solving our HFB equations.

```matlab
function data = HFB_Popov_2D_Trident(g,gd,T,ntotal,n0guess,nTguess,kmin,kmax,klength,ite

% HFB solver

kvec = linspace(kmin,kmax,klength);

n0 = n0guess;

if isempty(nTguess)
nT = zeros(size(kvec));
else
nT = nTguess;
end
nTold = nT;

maxiter = 200;
exitflag = 0;
dn0=1;
iter = 1;

tic
while (dn0>1e-16 && ~exitflag && iter<maxiter)

if iter<itertol
nTfrac = iter/itertol;
else
nTfrac = 1;
end
```

```
[wk,uk,vk,S11,S20,S20T0,nTnew,mu] = BdG(kvec,g,gd,n0,ntotal,nT,T);


if isreal(wk)
nT = nTfrac*nTnew+(1-nTfrac)*nTold;
nTtot = 2*pi/(2*pi)^2*trapz(kvec,kvec.*nT);
n0old = n0;
n0 = ntotal-nTtot;
n0-n0old;
dn0 = abs(n0-n0old);
else
n0=0;
exitflag = 1;
end


nTold = nT;
iter = iter+1;
t = toc;


end


% find superfluid fraction
nSF = SFfun(T,kvec,wk,ntotal);


% find local minimum in wk
dwk = gradient(wk,kvec);
wkind = min(find(gradient(dwk./abs(dwk))==1));
if isempty(wkind)
kroton = nan;
wkroton = nan;
else
kroton = kvec(wkind);
wkroton = wk(wkind);
```

```
    end

    fprintf('n0 = %G nSF = %G mu = %G\n',n0,nSF,mu)


    data.kroton = kroton;
    data.wkroton = wkroton;
    data.nSF = nSF;
    data.mu = mu;
    data.n0 = n0;
    data.nTtot = nTtot;
    data.nT = nT;
    data.n = ntotal;
    data.kvec = kvec;
    data.wk = wk;
    data.uk = uk;
    data.vk = vk;
    data.S11 = S11;
    data.S20 = S20;
    data.S20T0 = S20T0;
    data.iter = iter-1;
    data.exitflag = exitflag;
    data.t = t;


    % end of HFB.m


    function [wk,uk,vk,S11,S20,S20T0,nT,mu] = BdG(kvec,g,gd,n0,ntotal,nT,T)
    [S11,S20,S20T0,mu] = SEfun(kvec,g,gd,n0,ntotal,nT);
    Ak = kvec.^2/2-mu+S11;
    Bk = S20;


    wk = sqrt(Ak.^2-Bk.^2);
    uk = 1./sqrt(1-((wk-Ak)./Bk).^2);
```

```
vk = ((wk-Ak)./Bk)./sqrt(1-((wk-Ak)./Bk).^2);


if T>1e-6
fk = 1./(exp(wk/T)-1);
else
fk = 0;
end


nT = 1/2*(uk+vk).^2.*fk+vk.^2;


return


function [S11,S20,S20T0,mu] = SEfun(kvec,g,gd,n0,ntotal,nT)


thetavec = linspace(0,pi,128);
Vhn = (2*pi/(2*pi)^2)*trapz(kvec,kvec.*Vkfun(kvec,g,gd).*nT);
Vfn = zeros(size(kvec));
for ii = 1:length(kvec)
k = kvec(ii);
Vfvec = zeros(size(kvec));
for jj = 1:length(kvec)
q = kvec(jj);
Vfvec(jj) = 2*trapz(thetavec,Vkfun(sqrt(q.^2+k.^2-2*k*q*cos(thetavec)),g,gd));
end
Vfn(ii) = 1/(2*pi)^2*trapz(kvec,kvec.*Vfvec.*nT);
end


S11 = ntotal*Vkfun(0,g,gd)+n0*Vkfun(kvec,g,gd)+Vfn;
S20 = n0*Vkfun(kvec,g,gd);
S20T0 = n0*Vkfun(kvec,g,gd);
mu = ntotal*Vkfun(0,g,gd)+Vhn;
```

```
return


function Vk = Vkfun(kin,g,gd)


k = kin(:)/sqrt(2);
ks = k(k<=25);
kl = k(k>25);
vals = exp(ks.^2).*erfc(ks);
vall = 0.0763957775554068*exp(-0.049875774108393764*kl);
Vks = g+gd*(2-3*sqrt(pi)*ks.*vals);
Vkl = g+gd*(vall./kl-1);
Vk = [Vks;Vkl];
Vk = reshape(Vk,size(kin));


return


function nSF = SFfun(T,kvec,wk,ntotal)
nSF = ntotal+1/2*2*pi/(2*pi)^2*trapz(kvec,kvec.^3.*(1./(1-cosh(wk/T))))/2/T;
if nSF<0
nSF=0;
end
return
```